Problem 1

(a) (5 points) Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \\ 2 & 4 & 6 & 8 & 10 \\ 1 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

(b) (5 points) Use the result from (a) to compute the determinant of the matrix

$$B = \begin{bmatrix} 1 & 0 & 3 & 2 & 2 \\ 2 & 4 & 6 & 8 & 12 \\ 0 & 0 & 0 & 1 & 3 \\ 1 & 3 & 5 & 7 & 10 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}.$$

(a) A is a 5×5 block matrix with the lower left block of zeros (2×3).

Thus, $\det A = \det \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 1 & 0 & 3 \end{bmatrix} \det \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} =$

$$\begin{matrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{matrix}$$

$= (12+18-20-18)(6-2) =$

$= -8 \cdot 4 = -32$.

(b) B is obtained from A by the following:

1) Add column 1 to 5 (this does not change det);
2) Swap rows 4 and 5;
3) Swap rows 3 and 4;
4) Swap rows 1 and 4.

odd # of row swaps!

Thus, $\det B = -\det A = 32$.

Problem 2

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

(a) (5 points) Find an orthonormal basis for $Img(A)$ and perform the QR factorization for A;

(b) (2 points) What are the dimensions of $Img(A)$, $Img(A^t)$, $Ker(A)$ and $Ker(A^t)$?

(c) (3 points) Use (a) to write down the *orthogonal projection matrix* onto $Img(A)$.

(a) $\vec{u}_1 = \frac{1}{|\vec{v}_1|}\vec{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\vec{v}_2 \cdot \vec{u}_1 = \frac{3}{\sqrt{3}} = \sqrt{3}$.

$\vec{w}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} - \sqrt{3}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$

$\Rightarrow \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Since $\vec{v}_3$ is a multiple of $\vec{u}_2$, we stop here. We have:

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ -1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}.$$

Now, $\vec{u}_1 \cdot \vec{v}_1 = \sqrt{3}$, $\vec{u}_2 \cdot \vec{v}_2 = \sqrt{2} \Rightarrow$

$$\Rightarrow R = \Bigg[\underbrace{\begin{matrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{2} \end{matrix}}_{\text{this suffices, though!}} \;\Bigg|\; \begin{matrix} 0 \\ \frac{1}{\sqrt{2}} \end{matrix}\Bigg]$$

because $\vec{V}_3 = \frac{1}{2}(\vec{v}_2 - \vec{v}_1)$.

(b) Ans: 2, 2, 1, 1!

(c) Since $Img(A) = Img(Q) \Rightarrow P_S = QQ^t =$

$$= \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ -1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix}$$

Problem 3

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \text{ and the vectors } \mathbf{v} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(a) (5 points) Find the orthogonal projection of $\mathbf{v}$ onto $Ker(A^t)$;

(b) (5 points) Find the orthogonal projection of $\mathbf{w}$ onto $Ker(A)$.

(a) $\dim(\mathrm{Img}(A)) = 2 \Rightarrow \dim(\mathrm{Ker}(A^t)) = 1$.

Row-reduce $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \Rightarrow$

$\Rightarrow \begin{cases} x_3 = 0 \\ x_2 = s \\ x_1 = -s \end{cases} \Rightarrow \vec{a} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is a basis for $\mathrm{Ker}(A^t) = S$.

Thus $P_S = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} =$

$= \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow P_S \vec{v} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} =$

$= \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$ (one could see that if one notices that $\vec{v} \in \mathrm{Img}(A)$!).

(b) We know $\mathrm{Ker}(A) = (\mathrm{Img}(A^t))^{\perp}$.

$A^t = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$. So, a basis for $\mathrm{Img}(A^t)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$. Thus, $\mathrm{rank}(A^t) = \mathrm{rank}(A) = 2$.

Since $\mathrm{rank}(A) + \mathrm{null}(A) = \#\text{ columns} = 2 \Rightarrow$

$\Rightarrow \dim(\mathrm{Ker}(A)) = \mathrm{null}(A) = 0$. Thus, $\mathrm{Ker}(A) = 0$, so the projection of any vector onto it is 0.

Problem 4

Consider the point $P(0,1,1)$ and the line l of equations $x = y = 2z$ in $\mathbb{R}^3$.

(a) (6 points) Use the cross product to find the *normal equation* of the plane containing P and the line l;

(b) (2 points) Are the points $Q(-2,-2,-1)$ and $R(2,2,1)$ on l? Compute the area of the triangle ΔPQR;

(c) (2 points) Use (b) to find the distance between the point P and the line l.

(a) Could use $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ (direction of ℓ).

Thus, $\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{vmatrix} = -\vec{e}_1 + 2\vec{e}_2 - 2\vec{e}_3 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$.

Since the plane contains ℓ (which contains the origin), the normal eqn. is:

$$\boxed{x - 2y + 2z = 0}$$

(b) Yes! Both triplets satisfy $x = y = 2z$!

$\vec{PQ} = \begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix}$, $\vec{PR} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ -2 & -3 & -2 \\ 2 & 1 & 0 \end{vmatrix} = \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix} \Rightarrow \text{Area} = \frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{1}{2}\sqrt{36} = \boxed{3.}$

(c) $3 = \text{Area} = \frac{1}{2}|\vec{QR}| \cdot h \Rightarrow h = \frac{6}{|\vec{QR}|}$ $\Rightarrow \boxed{h = 1.}$

But $\vec{QR} = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}$, so $|\vec{QR}| = 6$