Problem 1

(a) (5 points) Is the sequence $a_n = (-1)^n/n$ bounded? Is it monotonic?

(b) (5 points) If $b_n = a_n^2$, find $\lim_{n\to\infty} b_n$. Is b_n monotonic?

(a) Note that $|a_n| = \frac{1}{n} \leq 1$ for all integers $n \geq 1$. So a_n is bounded.

Since $a_1 = -1$, $a_2 = \frac{1}{2}$, $a_3 = -\frac{1}{3}$ we deduce that $a_1 < a_2 > a_3$. This shows that a_n is <u>NOT</u> monotonic.

(b) So $b_n = a_n^2 = \left[\frac{(-1)^n}{n}\right]^2 = \frac{(-1)^{2n}}{n^2} = \frac{1}{n^2}$.

Obviously, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$.

Since $(n+1)^2 = n^2+2n+1 > n^2$ for all positive integers n, it follows that $\frac{1}{(n+1)^2} < \frac{1}{n^2}$, i.e. $b_{n+1} < b_n$.

Therefore, b_n is decreasing and so b_n is monotonic.

Problem 2

(*a*) (4 points) State the Mean Value Theorem.

(*b*) (6 points) Use Rolle's Theorem and the Intermediate Value Theorem to show that the equation

$$6x^5 + x^3 + 2x = 1000$$

has exactly one real root.

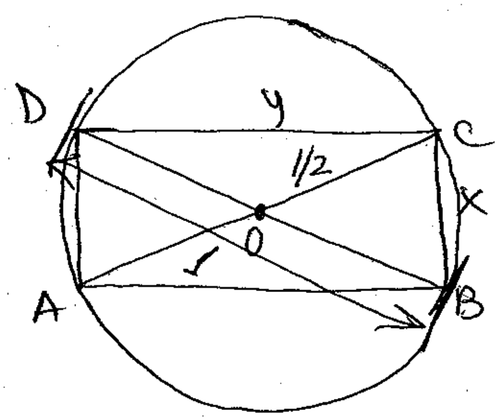
(a) If f is continuous on the closed, bounded interval $[a,b]$, differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

(b) Let $f(x) = 6x^5 + x^3 + 2x - 1000$.
As a polynomial function, f is differentiable on $(-\infty, \infty)$.
Note that $f(0) = -1000 < 0$.
Also, $f(10) = 600{,}000 + 1{,}020 - 1000 > 0$.
Thus, there exists a zero in the interval $(0, 10)$.
Assume that there are two distinct zeros say $a < b$.
Then, according to Rolle's Theorem, there should exist $c \in (a,b)$ such that $f'(c) = 0$.
But $f'(x) = 30x^4 + 3x^2 + 2 > 2$ for all x,
so such a c cannot exist.

Problem 3

(10 points) What is the largest possible area for a rectangle inscribed in a circle of radius 1/2?



The area is:

$S = xy$. But, according to the Pythagorean Theorem we have:

$x^2 + y^2 = 1^2 = 1$.

Thus $y^2 = 1 - x^2$. Note that maximizing $S = xy$ is equivalent to maximizing x^2y^2 (since all lengths involved should be positive).

Thus, consider $f(x) = x^2(1-x^2)$ for $x \in [0,1]$.

We have $f'(x) = 2x - 4x^3 = 2x(1-2x^2)$.

Therefore, there are three critical numbers:

$x_1 = 0$, $x_{2,3} = \pm\sqrt{\frac{1}{2}}$.

The values ~~that~~ the end points are 0.

The second derivative is $f''(x) = 2 - 12x^2 = 2(1-6x^2)$. Thus, $f''(-\frac{1}{\sqrt{2}}) = f''(\frac{1}{\sqrt{2}}) = -4 < 0$.

Since $-\frac{1}{\sqrt{2}}$ is negative (so not admissible), we infer that the maximum is attained at $x = \frac{1}{\sqrt{2}}$

Thus, the max area occurs when $x = y = \frac{1}{\sqrt{2}}$ and it is $S = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$.

Problem 4

(20 points) Sketch the graph of

$$f(x) = \begin{cases} -\frac{x}{x^2+1} & \text{if } x \leq 0 \\ \frac{1}{x-1} & \text{if } x > 0. \end{cases}$$

by first identifying as many points of interest and features of the graph as possible.

1. The domain of f is $(-\infty, 1) \cup (1, \infty)$ because f is not defined at $x=1$. Since $f(x) \to -\infty$ as $x \to 1^-$ and $f(x) \to +\infty$ as $x \to 1^+$, it follows that $x=1$ is a Vertical Asymptote. Then: $f(x) \to 0$ as $x \to \pm\infty$. Thus, $y=0$ is a Horizontal Asymptote.

2. Since $f(0)=0$ and $f(x)=0$ implies $x=0$, we conclude that $(0,0)$ is the only intercept (note that the point $(0,-1)$ is NOT on the graph).

3. No symmetry, no periodicity!

4. Calculate $f'(x) = \begin{cases} \frac{x^2-1}{(x^2+1)^2} & \text{if } x \leq 0 \\ -\frac{1}{(x-1)^2} & \text{if } x > 0, x \neq 1. \end{cases}$

The critical numbers could be $x_{1,2} = \pm 1$. But f is not even defined at $x=1$, so we only have one critical number $x=-1$.

5. Calculate $f''(x) = \begin{cases} \frac{2x(3-x^2)}{(x^2+1)^3} & \text{if } x \leq 0 \\ \frac{2}{(x-1)^3} & \text{if } x > 0, x \neq 1. \end{cases}$

If $f''(x) = 0$, we obtain $x_1 = 0$ and $x_2 = -\sqrt{3}$.

6.

x	$-\infty$	$-\sqrt{3}$	-1	0	1	$+\infty$
f'	+ +	+	− −	− −	/	− − − −
f''	+ +	− • −	− −	− −	/	+ + + +
f	↗	↗	↘	0 ↘	/	↘

7. Thus, the graph looks like: