Problem 1

(*a*) (4 points) Use the fact that

$$\ln(a^y) = y \ln a \text{ for all } a > 0 \text{ and all real } y$$

to prove that

$$(e^x)^y = e^{xy}.$$

(*b*) (6 points) Let $f(x) = x^{x^2}$ for $x > 0$. Calculate $f'(1)$.

(a) Let $a = e^x$ and $y = y$.

Then $a^y = (e^x)^y$ and so:

$\ln a^y = \ln (e^x)^y = y \ln e^x \implies$

But $\ln e^x = x$

$\implies \ln(e^x)^y = yx = xy = \ln(e^{xy})$.

Thus, $(e^x)^y = e^{xy}$.

(b) Since $x = e^{\ln x}$, we have:

$f(x) = (e^{\ln x})^{x^2} = e^{x^2 \ln x}$.

Thus: $\frac{d}{dx} f(x) = \frac{d}{dx}\left(e^{x^2 \ln x}\right) = e^{x^2 \ln x} \frac{d}{dx}(x^2 \ln x)$.

$= x^{x^2}\left(2x \ln x + x^2 \frac{1}{x}\right) = x^{x^2+1}(2\ln x + 1)$.

Therefore:

$f'(1) = 1^2(2\underbrace{\ln 1}_{=0} + 1) = 1.$

Problem 2

(a) (5 points) Use partial fraction decomposition to calculate

$$\int \frac{dx}{x^3 + x^2}.$$

(b) (5 points) Use an appropriate trigonometric substitution to show that

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C \text{ for } |x| < 1.$$

(a) We have:

$$\frac{1}{x^3+x^2} = \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \iff$$

$$\iff 1 = Ax(x+1) + B(x+1) + Cx^2.$$

If $x = 0 \Rightarrow B = 1$

If $x = -1 \Rightarrow C = 1$

If $x = 1 \Rightarrow 2A + 2B + C = 2A + 3 = 1 \Rightarrow A = -1.$

Thus $\int \frac{1}{x^3+x^2} dx = -\int \frac{1}{x} dx + \int \frac{1}{x^2} dx + \int \frac{1}{x+1} dx =$

$= -\ln|x| - \frac{1}{x} + \ln|x+1| + C.$

(b) Let $x = \cancel{\sin u}$ $\sin u \Rightarrow dx = \cos u \, du.$

and $\sqrt{1-x^2} = \sqrt{1-\sin^2 u} = \cos u.$

Thus: $\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cancel{\cos u} du}{\cancel{\cos u}} = \int du = u + C$

But $x = \sin u \Rightarrow u = \sin^{-1} x \Rightarrow$

$\Rightarrow \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C.$

Problem 3

(*a*) (4 points) Use integration by parts to calculate

$$\int \ln x dx.$$

(*b*) (6 points) Use integration by parts and the integral from (*a*) to compute

$$\int (\ln x)^2 dx.$$

Hint for (b): Let $u = (\ln x)^2$ *and* $dv = dx$.

(a) Let $\left.\begin{matrix} u = \ln x \\ dv = dx \end{matrix}\right| \Rightarrow \begin{matrix} du = \frac{1}{x} dx \\ v = x \end{matrix}$.

Thus $\int \ln x \, dx = uv - \int v du =$

$= x \ln x - \int x \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C$

(b) Let $\left.\begin{matrix} u = (\ln x)^2 \\ dv = dx \end{matrix}\right| \Rightarrow \begin{matrix} du = 2 \frac{\ln x}{x} dx \\ v = x \end{matrix}$.

Thus: $\int (\ln x)^2 dx = uv - \int v du =$

$= x(\ln x)^2 - \int \cancel{x} \frac{2 \ln x}{\cancel{x}} dx =$

$= x(\ln x)^2 - 2 \int \ln x dx \underset{=}{\overset{\text{using (a)}}{=}}$

$= x(\ln x)^2 - 2x \ln x + 2x + C.$

Problem 4

(a) (5 points) Calculate

$$\int_0^{\pi/2} \sin^2 x \cos^3 x dx.$$

(b) (5 points) Compute, by an appropriate substitution (trigonometric?),

$$\int_{-1}^{1} \sqrt{1-x^2}dx.$$

(a) Let $u = \sin x$. Note that:

$$\int_0^{\pi/2} \sin^2 x \cos^3 x dx = \int_0^{\pi/2} \sin^2 x \cos^2 x \cos x dx =$$

$$= \int_0^{\pi/2} \sin^2 x (1-\sin^2 x)(\sin x)' dx = \int_0^1 u^2(1-u^2)du =$$

$$= \left(\frac{u^3}{3} - \frac{u^5}{5} + C\right)\Big|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}.$$

(b) Let $x = \sin u \Rightarrow dx = \cos u \, du$

Thus: $$\int_{-1}^{1} \sqrt{1-x^2} dx = \int_{-\pi/2}^{\pi/2} \cos u \cos u \, du =$$

$$= \int_{-\pi/2}^{\pi/2} \cos^2 u \, du = \int_{-\pi/2}^{\pi/2} \frac{1-\cos 2u}{2} du =$$

$$= \frac{\pi}{2} - \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos 2u \, d(2u) = \frac{\pi}{2} - \frac{1}{4}\int_{-\pi}^{\pi} \cos v \, dv =$$

$$= \frac{\pi}{2} - \frac{1}{4} \sin v \Big|_{-\pi}^{\pi} = \frac{\pi}{2}.$$