Problem 1

(*a*) (5 points) Compute the determinant of the matrix

$$A = \begin{bmatrix} a & 2 & 0 & 0 \\ 2 & a & 2 & 0 \\ 0 & 2 & a & 2 \\ 0 & 0 & 2 & a \end{bmatrix}.$$

(*b*) (2 points) For which values of a is A invertible?

(*c*) (3 points) Use the result from (*a*) and some properties of the determinant function to compute the determinant of the matrix

$$B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

(a) Expand by first row: block matrix

$$\det A = a \begin{vmatrix} a & 2 & 0 \\ 2 & a & 2 \\ 0 & 2 & a \end{vmatrix} - 2 \left| \begin{array}{c|cc} 2 & 2 & 0 \\ \hline 0 & a & 2 \\ 0 & 2 & a \end{array} \right| =$$

$$\begin{matrix} a & 2 & 0 \\ 2 & a & 2 \end{matrix}$$

$$= a(a^3 - 8a) - 4(a^2 - 4) = a^4 - 12a^3 + 16.$$

(b) Let $x = a^2$. Solve $x^2 - 12x + 16 = 0$;

$$x_{1,2} = \frac{12 \pm \sqrt{144 - 64}}{2} = \frac{12 \pm 4\sqrt{5}}{2} = 6 \pm 2\sqrt{5}.$$

So, $a^2 = 6 \pm 2\sqrt{5} \Longleftrightarrow a_{1,2,3,4} = \pm\sqrt{6 \pm 2\sqrt{5}}$

$= \pm\sqrt{(\sqrt{5} \pm 1)^2} = \pm(\sqrt{5} \pm 1)$.

Thus, A is invertible if $a \neq \pm(\sqrt{5} \pm 1)$ (or $\mp$).

(c) Note that $16B = \begin{bmatrix} 0 & 0 & 2 & 4 \\ 0 & 2 & 4 & 2 \\ 2 & 4 & 2 & 0 \\ 4 & 2 & 0 & 0 \end{bmatrix}$. If we swap rows 1 and 4, then 2 and 3, we get back the matrix A with $a = 4$.

So, $\det B = (-1)^2 \det A = 4^4 - 12 \cdot 4^3 + 16 = 3{,}912$.

Problem 2

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

(*a*) (6 points) Find an orthonormal basis for $Img(A)$ and perform the QR factorization for A;

(*b*) (2 points) What are the dimensions of $Img(A)$ and $Img(A^t)$?

(*c*) (2 points) Are the rows of A linearly independent? How about its columns?

a) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$

$Img(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$

Let $v_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$

$u_1 = \dfrac{v_1}{|v_1|} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$

$w_2 = v_2 - (u_1 \cdot v_2) u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix} \Rightarrow$

$u_2 = \dfrac{w_2}{|w_2|} = \dfrac{1}{\sqrt{3/2}} \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix} \Rightarrow$

$Q = [u_1 \ u_2].$

$R = \begin{bmatrix} u_1 \cdot v_1 & u_1 \cdot v_2 & u_1 \cdot v_3 \\ u_2 \cdot v_1 & u_2 \cdot v_2 & u_2 \cdot v_3 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & (3/2)/\sqrt{3/2} & 3/\sqrt{3/2} \end{bmatrix}$

b) $\dim(Img(A)) = \dim(Img(A^t)) = 2$

c) Neither the columns or rows of A are linearly independent.

Problem ~~4~~ 3

Consider the vectors and the matrix below

$$\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad M = \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}.$$

(*a*) (4 points) Compute the cross product $\overset{v}{\cancel{\mathbf{a}}} \times \overset{w}{\mathbf{b}}$ and use it to find the equation of the plane spanned by $\mathbf{v}$ and $\mathbf{w}$;

(*b*) (4 points) Use row reduction to find a nonzero vector in $Ker(M)$ and use this vector to find the equation of the plane spanned by $\mathbf{v}$ and $\mathbf{w}$. Do the results from (*a*) and (*b*) agree?

(*c*) (2 points) What is the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$?

(a) $\vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 3 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \vec{e}_1 \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} - \vec{e}_2 \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$

$+ \vec{e}_3 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -7\vec{e}_1 + 3\vec{e}_2 - \vec{e}_3 = \begin{bmatrix} -7 \\ 3 \\ -1 \end{bmatrix}.$

Thus, the orthonormal equation of the plane is $-7x_1 + 3x_2 - x_3 = 0$.

(b) $M = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & -1 \end{bmatrix} \to \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right] \to$

$\to \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_2 + 3x_3 = 0 \\ x_1 + 3x_2 + 2x_3 = 0 \end{array} \Rightarrow$

$\Rightarrow x_3 = t, \; x_2 = -3t, \; x_1 = 9t - 2t = 7t.$

$\Rightarrow \vec{v} = \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \Rightarrow 7x_1 - 3x_2 + x_3 = 0$. Yes! They agree!

(c) $A = |\vec{v} \times \vec{w}| = \left| \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix} \right| = \sqrt{49+9+1} = \sqrt{59}.$

Problem 4

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and the vector } \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(a) (5 points) Find the orthogonal projection of $\mathbf{v}$ onto $Img(A)$:

(b) (5 points) Find the orthogonal projection of $-\mathbf{v}$ onto $Ker(A^t)$.

(a) Since $\operatorname{rank}(A) = 2$, we know $\dim Ker(A^t) = 1$.
Thus, it is easier to find $P_{S^\perp}$ first.
Note that $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \in Ker(A^t) = S^\perp$ and,
since $S^\perp$ is one-dimensional $\Rightarrow$

$$P_{S^\perp} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} =$$

$$= \frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow P_S = I_3 - P_{S^\perp} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Thus $\operatorname{proj}_{Img(A)} \vec{v} = P_S \vec{v} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$= \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Also, $\operatorname{proj}_{Ker(A^t)}(-\vec{v}) = \frac{1}{3} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$

$$= \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}.$$