Problem 1

(*a*) (5 points) Find the interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}(x+1)^n.$$

(*b*) (5 points) Let $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$. What is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{y}$?

(a) Look at $\left| \dfrac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} \dfrac{(x+1)^{n+1}}{(x+1)^n} \right| = 2\left(\frac{n}{n+1}\right)^2 |x+1|$

$\xrightarrow[n\to\infty]{} 2|x+1|$. Need $2|x+1| < 1 \Rightarrow$

$\Rightarrow |x+1| < \frac{1}{2} \Rightarrow \left(-\frac{3}{2}, -\frac{1}{2}\right)$ is part of IC.

At $x = -\frac{3}{2}$, $\sum_{n=1}^{\infty} \frac{2^n}{n^2}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is cv. (abs. cv.).

At $x = -\frac{1}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ is cv.

So, IC $= [-3/2, -1/2]$.

(b) We have $\vec{x}_{\parallel} = (\vec{x}\cdot\vec{u})\vec{u}$ where $\vec{u} = \frac{\vec{y}}{|\vec{y}|}$.

So, $\vec{x}_{\parallel} = \dfrac{(\vec{x}\cdot\vec{y})\vec{y}}{|\vec{y}|^2} = \dfrac{7}{25}\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 21/25 \\ 28/25 \end{bmatrix}$.

Problem 2

(*a*) (5 points) Expand $f(x) = x^3 \sin(2x)$ in powers of x. What is the 99^{th} derivative of f at 0?

(*b*) (5 points) Prove that

$$\int_0^1 \cos(t^2)\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(4n+1)}.$$

Hint: Use integration of power series to write $\int_0^x \cos(t^2)\mathrm{d}t$ *as a power series and then plug in the appropriate value for* x.

(a) Since $\sin y = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}$ for all $y \in \mathbb{R}$,

we have: $\sin(2x) = \sum_{k=0}^{\infty} (-1)^k \frac{(2x)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} x^{2k+1}$.

Thus, $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} x^{2k+4}$. Since there are only even exponents in the expansion, we conclude that $f^{(99)}(0) = 0$.

(b) We have $\cos(t^2) = \sum_{k=0}^{\infty} (-1)^k \frac{(t^2)^{2k}}{(2k)!} =$

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{4k}$.

Thus, $\int_0^x \cos(t^2)dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \int_0^x t^{4k} dt = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{(2k)!(4k+1)}$

Let $x = 1$ to conclude.

Problem 3

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + 2y \end{bmatrix}.$$

(a) (2 points) Write down the matrix of f.

(b) (5 points) Use (a) to write down the transformation $f \circ f \circ f$.

(c) (3 points) Is f invertible? If so, find its inverse.

(a) Since $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, we get:

$A_f = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$.

(b) We have $A_{f\circ f\circ f} = A_f A_f A_f = A_f^3$.

$= \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} =$

$= \begin{bmatrix} -3 & -6 \\ 6 & 3 \end{bmatrix}$. Thus,

$(f\circ f\circ f)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -3x - 6y \\ 6x + 3y \end{bmatrix}$.

(c) Since $1 \cdot 2 - (-1) \cdot 1 = 2 + 1 = 3 \neq 0 \Rightarrow$

$\Rightarrow A_f$ is invertible and $A_f^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$.

$\Rightarrow f^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 2x + y \\ -x + y \end{bmatrix}$.

Problem 4

Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}.$$

(a) (5 points) Compute $(AB)_{2,1} - (B^tA)_{2,3}$ without computing the whole matrix products AB and B^tA.

(b) (5 points) Is it possible to add the matrices B^t and B^tA? If so, write down their sum.

(a) To compute $(AB)_{21}$ we compute $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 2.$

Since $B^t = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 1 & 3 \end{bmatrix}$, we need $\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0.$

So, $(AB)_{2,1} - (B^tA)_{2,3} = 2 - 0 = 2.$

(b) Since $B^t \in \mathcal{M}_{2\times3}$ and $A \in \mathcal{M}_{3\times3} \Rightarrow$

$\Rightarrow B^tA \in \mathcal{M}_{2\times3}$. So, we can add

$$B^t + B^tA = B^t(I_3 + A) =$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 7 \\ 1 & 11 & 3 \end{bmatrix}.$$