Problem 1

1p (a) (5 points) Find the sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1}3^{-n}$.

1p (b) (5 points) Compute $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{4x}$.

1p (c) (5 points) Write down the 5^{th} Taylor polynomial associated to the function $f(x) = x^3 + \cos(x^2)$.

(a) $$\sum_{n=1}^{\infty}(-1)^{n+1}3^{-n} = (-1)\sum_{n=1}^{\infty}(-1)^n 3^{-n} = (-1)\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n =$$

$$= -\left[\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n - 1\right] = 1 - \frac{1}{1-\left(-\frac{1}{3}\right)} = 1 - \frac{3}{4} = \frac{1}{4}.$$

(b) Take the log to see that:

$$\ln\left(1-\frac{1}{x}\right)^{4x} = 4x\ln\frac{x-1}{x} = 4\,\frac{\ln\frac{x-1}{x}}{\frac{1}{x}}.$$

So, as $x\to\infty$, we are in the $\frac{0}{0}$ case.

To apply L'Hôpital's Rule, look at:

$$\frac{\left(\ln\frac{x-1}{x}\right)'}{\left(\frac{1}{x}\right)'} = \frac{\frac{1}{x-1}-\frac{1}{x}}{-\frac{1}{x^2}} = -\frac{x^2}{x^2-x} \xrightarrow[\text{as } x\to\infty]{} -1.$$

Thus, $\lim_{x\to\infty} \ldots = e^{-4}$.

(c) Since $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, we see that $\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots$

So, $P_5(x) = -\frac{1}{2}x^4 + x^3 + 1.$

Problem 2

1p (a) (5 points) Is $\sum_{k=0}^{\infty} \frac{2k+1}{\sqrt{k^4+2}}$ convergent? Justify!

1p (b) (5 points) Is $\sum_{k=0}^{\infty} k^2 e^{-k^3}$ convergent? Justify!

1p (c) (5 points) Prove that $\sum_{k=0} \frac{k!(2k)!}{(3k)!}$ is convergent.

(a) Since $\lim_{k\to\infty} \frac{k(2k+1)}{\sqrt{k^4+2}} = \lim_{k\to\infty} \frac{2k^2+k}{\sqrt{k^4+2}} = 1$, we deduce (Limit Comparison) that our series behaves like the harmonic $\sum \frac{1}{k}$. So, it is divergent.

(b) Root test: $\left(k^2 e^{-k^3}\right)^{1/k} = k^{2/k} e^{-k^2}$.

Since $k^{2/k} \xrightarrow[k\to\infty]{} 1$ and $e^{-k^2} \xrightarrow[k\to\infty]{} 0 \Rightarrow$ $k^{2/k} e^{-k^2} \xrightarrow[k\to\infty]{} 0$. So, we have convergence.

(c) Ratio Test:

$$\frac{a_{k+1}}{a_k} = \frac{(3k)!}{k!(2k)!} \frac{(k+1)!(2k+2)!}{(3k+3)!}$$

$$= \frac{(k+1)(2k+1)(2k+2)}{(3k+1)(3k+2)(3k+3)} \xrightarrow[k\to\infty]{} \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{27} < 1.$$

So, indeed, the series is convergent.

Problem 3

1p (a) (5 points) Is $\sum_{k=1}^{\infty}(\sqrt{k+1}-\sqrt{k})^{k/2}$ convergent? Justify!

1p (b) (5 points) Is $\sum_{k=0}^{\infty}(-1)^k\frac{k}{k^2+1}$ absolutely convergent? Is it convergent? Justify!

1p (c) (5 points) Does the improper integral $\int_2^{\infty}\frac{\sqrt{x}}{x^3-1}dx$ converge? Justify!

(a) Take $a_k=(\sqrt{k+1}-\sqrt{k})^{k/2}$ and apply the Root Test:

$$a_k^{1/k}=(\sqrt{k+1}-\sqrt{k})^{1/2}=\left(\frac{(k+1)-k}{\sqrt{k+1}+\sqrt{k}}\right)^{1/2}=$$

$$=\left(\frac{1}{\sqrt{k+1}+\sqrt{k}}\right)^{1/2}\xrightarrow[k\to\infty]{}0.$$ Yes, the series is convergent.

(b) No, by Limit Comparison with $\sum\frac{1}{k}$. Indeed, $\lim_{k\to\infty}\frac{k^2}{k^2+1}=1$.

Yes, because $\lim_{k\to\infty}\frac{k}{k^2+1}=0$ and the function $f(x)=\frac{x}{x^2+1}$ is decreasing on $[1,\infty)$. Indeed, $f'(x)=\frac{x^2+1-x\cdot 2x}{x^2+1}=\frac{1-x^2}{x^2+1}<0$ on $(1,\infty)$. So, we have a convergent alternating series.

(c) We have $x^3-1>\frac{x^3}{2}\iff x^3>2$ on $[2,\infty)$.

So, $\frac{\sqrt{x}}{x^3-1}<\frac{2\sqrt{x}}{x^3}=\frac{2}{x^{5/2}}$. Since $5/2>1$, we get convergence by Comparison with $2\int_2^{\infty}\frac{1}{x^{5/2}}dx$.