

The Exponential Function

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0. Introduction.

In these notes we

1. Review the basics of the exponential function a^n where $a > 0$, n a natural number, see how to extend it to rational, then irrational, and finally negative exponents.
2. Show how to calculate its derivative.
3. Introduce the important number e , the so-called base of the natural logarithm, as the limit of a sequence.
4. See how this limit (and hence the exponential function) arises in the context of compound interest.

We will use the notion of convergence of a sequence of real numbers, and the Least Upper Bound Axiom for the real numbers. If you are not familiar with these topics, you should look them up in your calculus text. In the book *Calculus, one and several variables, ninth edition*, by Salas, Hille, and Etgen, which we are currently using at Georgia Tech, this material is covered in Sections 10.1 through 10.3, beginning on page 585.

1. The exponential function.

When $a > 0$, and $n = 1, 2, 3, \dots$ is a natural number, the exponential

$$a^n = \underbrace{a a \cdots a}_{n \text{ times}}$$

is simply the product of a with itself n times. For $a, b > 0$, this function satisfies the usual rules

- (1)
- a. $a^{n+m} = a^n a^m$,
 - b. $(a^n)^m = a^{nm}$,
 - c. $a^n b^n = (ab)^n$,
 - d. If $a \neq 1$, then $a^n = a^m$ if and only if $n = m$,
 - e. If $a > 1$, then a^n is an increasing function of n ,
 - f. If $a < 1$, then a^n is a decreasing function of n .

On the other hand, if we regard n as fixed, and $a > 0$ as the variable, a^n is an increasing function of a . Using this observation and the Least Upper Bound Axiom¹, we see that for each natural number n and each real number $a > 0$, there is a unique positive real number b such that

$$b^n = a.$$

We call b the n -th root of a , and denote it by $\sqrt[n]{a}$. Specifically,

$$\sqrt[n]{a} = \text{Least Upper Bound of the set } \{x : x^n \leq a\}.$$

Next for a *rational* exponent, that is, one which is the *ratio* of two non-negative natural numbers as

$$a^{p/q} = (\sqrt[q]{a})^p = \sqrt[q]{a^p},$$

and we leave it as an exercise for the reader to convince itself that the rules (1) still hold when the exponents are allowed to be rational numbers.

For real numbers r which are not rational, such as $\sqrt{2}$ or π , we again look to the increasing nature of the exponential function (*le* above) and the Least Upper Bound Axiom and define, for $a > 1$

$$a^r = \text{Least Upper Bound of the set } \{a^s : s < r, s \text{ rational.}\}.$$

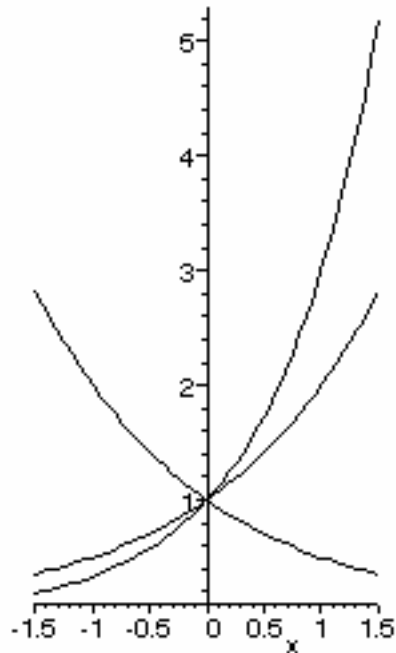
¹ Recall that the *Least Upper Bound Axiom* states that every non-empty set of real numbers which has an upper bound has a least upper bound. As an example, the numbers 5, π , and Avogadro's constant $6.02 \cdot 10^{23}$, are examples of upper bounds for the set $A = \{x : x = 1 - \mathcal{V}_n, n = 1, 2, 3, \dots\} = \{\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \dots\}$. The *least upper bound* for this set is 1, since no smaller number is an upper bound, and every other upper bound is at least as large as 1. Please see your calculus text for further explanation.

We leave it to the reader to figure out how to define a^r for $0 < a < 1$, and finally when the exponent r is negative ($r < 0$), we define

$$a^r = \frac{1}{a^{-r}}.$$

Notice that since $r < 0$, the right hand side of this equation is already defined. Again, please convince yourself that the rules in (1) are valid when the exponents are any real numbers.

Sketched below are the graphs of exponential functions with bases $a = \frac{1}{2}$, 2 and 3. Which is which?



2. The derivative of the exponential function.

From the above picture it is intuitively clear that exponential functions have well defined tangent lines, and hence, the derivative of the exponential--the slope of this tangent line--exists. Let's try to calculate it.

$$\begin{aligned}
 \frac{d}{dx} a^x &= \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{a^x a^h - a^x}{h} \\
 &= a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h} \\
 &= a^x \lim_{h \rightarrow 0} \frac{a^h - a^0}{h}.
 \end{aligned}$$

Now $\lim_{h \rightarrow 0} \frac{a^h - 1}{h}$ is just a constant. In fact, it's the derivative of a^x evaluated at $a = 0$.

That is, the derivative of the exponential function is

$$\frac{d}{dx} a^x = k_a a^x,$$

a constant multiple of itself, and that constant (which depends on the base a) is the slope at the point $x = 0$.

From the graph above, that slope at 0 is positive and large for some a , and negative for other a . Perhaps the most natural is that value of a for which the derivative at 0 is 1. This is the so called *base of the natural logarithm*, denoted by e . If you draw the above graph carefully, you'll see that the derivative of 2^x at 0 is less than 1, and the derivative of 3^x at 0 exceeds 1. Thus $2 < e < 3$.

Exercises.

1. Carefully sketch (by hand or by computer) the graphs of

$$f(x) = 2^x \quad \text{and} \quad g(x) = 3^x,$$

using equal scales on the axes. Carefully sketch the tangent lines at $x = 0$, and convince yourself that $f'(0) < 1 < g'(0)$. Deduce that $2 < e < 3$.

2. Sketch the graphs and calculate the first and second derivatives.

a. $y = e^{-2x} \cos(4x)$

b. $y = e^{3x} \sin(2x)$

c. $\sinh(x) = \frac{e^x - e^{-x}}{2}$ (the *hyperbolic sine*)

d. $\cosh(x) = \frac{e^x + e^{-x}}{2}$ (the *hyperbolic cosine*)

3. The number e .

As outlined above, we define e to be that number for which

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1,$$

and by exercise 1, it is plausible that there is such a positive number, and that it lies somewhere between 2 and 3.

Using the symbol \approx to mean "approximately equal," we have that for values of h near 0,

$$\frac{e^h - 1}{h} \approx 1,$$

so

$$\begin{aligned} e^h &\approx 1 + h, \\ e &\approx (1 + h)^{\frac{1}{h}}, \end{aligned}$$

so we expect that we may calculate e as

$$e = \lim_{h \rightarrow 0} (1 + h)^{\frac{1}{h}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

the limit of a sequence. You might (incorrectly) think that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ should be 1, since the terms $1 + \frac{1}{n}$ become close to 1, and 1 to any power is 1. You might (also incorrectly) think that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ should be infinite since each term $1 + \frac{1}{n}$ is strictly bigger than 1, and powers increasing without bound tend to infinity. Remember, we saw in Exercise 1 that $2 < e < 3$, so these arguments must both be wrong. The point is that *the base $1 + \frac{1}{n}$ is decreasing towards 1 at the same time as the exponent is increasing to ∞* . These actions are not independent, as assumed in the specious arguments above.

Here is a proof that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ actually does exist. The value, of course, is the number we call e . In our argument, we will use Bernoulli's inequality, which states that for $a > -1$, $a \neq 0$, $(1 + a)^k > 1 + ka$.

Define two sequences $s_n = \left(1 + \frac{1}{n}\right)^n$, $t_n = \left(1 + \frac{1}{n}\right)^{n+1}$. We will show that

- a. the s_n form a strictly increasing sequence,
- b. the t_n form a strictly decreasing sequence,
- c. $s_n < t_n$ for each n .

Consequently $\{s_n\}$ and $\{t_n\}$ are bounded, monotone sequences, and thus have limits. Since $t_n = s_n \left(1 + \frac{1}{n}\right)$, their limits are the same -- that number we call e , and since $s_n < e < t_n$ we can calculate s_n and t_n and thus approximate e to as many decimal places as we choose.

To show that the s_n form a strictly increasing sequence, we note

$$\begin{aligned}
 \frac{s_{n+1}}{s_n} &= \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} \\
 &= \left(1 + \frac{1}{n}\right) \left(\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}}\right)^{n+1} \\
 &= \frac{n+1}{n} \left(\frac{n^2 + 2n}{(n+1)^2}\right)^{n+1} \\
 &= \frac{n+1}{n} \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \\
 &> \frac{n+1}{n} \left(1 - \frac{n+1}{(n+1)^2}\right) \quad (\text{by Bernoulli's Inequality with } a = -\frac{1}{(n+1)^2}, \text{ and } k = n+1) \\
 &= 1.
 \end{aligned}$$

Thus $s_{n+1} > s_n$ and the sequence is increasing. We leave the proofs of statements b and c to the reader.

Exercises.

3. Bernoulli's inequality can be proved by *mathematical induction*. Use mathematical induction (look it up in your calculus text if is not familiar to you) to prove Bernoulli's inequality.
4. Use the inequalities $s_n < e < t_n$ to approximate e to one decimal place. What's the smallest n you can use?

5. By changing variables, show that for any r , $e^r = \lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n$. You may assume that the limits $\lim_{h \rightarrow 0} (1 + h)^{1/h}$ (h real) and $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ (n a natural number) are both equal to e .

4. Compound Interest.

The basic formula for *simple* interest is $i = Prt$. That is, interest equals principal times rate times time. For example, $\$P$, invested for t years at an interest rate of r per cent per year (expressed as a decimal), yields interest i . Thus $\$1,000$ invested at 4% for 2 years yields interest of $(1000)(.04)(2) = \$80$, and the investment is then worth $\$1080$.

More commonly, interest is *compounded*. If, for example, interest is compounded annually, one begins year 1 with $\$P$ and ends year 1 with $\$(P + Pi) = P(1 + i)$. This principal is invested for all of the second year, at the end of which the investment's value is

$$(P + Pi) + (P + Pi)i = P(1 + i)^2$$

In general, the value of the investment after t years is

$$P(1+i)^t.$$

If our $\$1,000$ is invested at 4% for two years compounded annually, the investment grows to a value of $\$1081.60$, so compounding is clearly in the investor's favor.

If interest is compounded *quarterly* (four times per year), then the annual interest rate r is a *quarterly rate* of $r/4$, and the analysis above shows that after k quarters, the value of the investment is

$$P\left(1 + \frac{r}{4}\right)^k.$$

Thus $\$1,000$ invested at 4% compounded quarterly for two years (8 quarters) grows to a value of

$$1000\left(1 + \frac{.04}{4}\right)^{(4)(2)} = 1000\left(\left(1 + \frac{.04}{4}\right)^4\right)^2 \approx 1082.86.$$

More generally, if interest is compounded m times per year, the value of the investment after t years is

$$P\left(1 + \frac{r}{m}\right)^{mt},$$

and in what is called *continuous compounding*, (the limit as m tends to ∞), the value of the investment after t years is

$$Pe^{rt}$$

by the result of exercise 4.

Exercises.

6. On the same set of axes, plot the values of an investment of \$1, invested at 6% per year compounded annually, quarterly, monthly, daily, and continuously, as a function of time t measured in years.
7. At age 20, you wish to invest \$ P , to be compounded continuously at 6% per year, in order to be able to pay for your childrens' education, beginning when you are 50. Make reasonable assumptions as to the cost of education (no Hope scholarship!), number of children (positive), and calculate P . Do not take income tax into account.
8. Suppose instead that at age 20, you begin investing \$ Q at the beginning of each year until you are 50. Using the same cost of education, number of children, and interest rate as in problem 6, calculate the amount Q you must set aside in order to pay for your childrens' education. Again, ignore taxes. (Hint: Think of doing problem 6 at ages 20, 21, 22, ...)

References.

Johnson, Kiokemeister, *Calculus with analytic geometry, second edition*, Allyn and Bacon, Inc., 1963.

Salas, Hille, Etgen, *Calculus: one and several variables, ninth edition*, John Wiley and Sons, Inc., 2003.