Name________________________________ 20 Aprl 2006

Teaching Assistant_____________________ Hour Test 3
Math 1502J Andrew

Instructions: 1. Closed book.
2. Show your work and explain your answers and reasoning.
3. Calculators may be used, but pay particular attention to instruction 2. **To receive credit, you must show your work.** Unexplained answers, and answers not supported by the work you show, will not receive credit.
4. Express your answers in simplified form.

1. (25) Find a basis for the image and a basis for the kernel of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & 4 & 3 & 0 & 4 \\ 0 & 0 & -1 & 2 & 2 \\ 1 & 2 & 2 & 0 & 3 \end{pmatrix}.$$

Compute the dimension of the image and the dimension of the kernel.

2. (25) Find the QR factorization of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 2 & 0 \\ 1 & 2 & 4 \end{pmatrix}$ by applying the Gram-Schmidt process to the columns of **A**.

3. (25) A matrix **A** has QR factorization $\mathbf{A} = \mathbf{Q}\mathbf{R} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & -1/2 \\ -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$.

Use this QR factorization to find

a. The least-squares solution to $\mathbf{A}\,\mathbf{x} = \begin{pmatrix} 3 \\ -3 \\ -1 \\ 7 \end{pmatrix}$

b. The matrix for the orthogonal projection onto the image of **A**.

4. (25) Find a diagonal matrix **D** and an invertible matrix **V** such that
$\mathbf{A} = \begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ and use them to calculate $\mathbf{A}^8$.

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Answers.

1. A basis for the image can be found by row reducing **A** to identify the pivot columns and then using the pivot columns of **A**. Thus a basis is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \right\}$, and the image of **A** has dimension 3. A basis for the kernel of **A** is found from the general solution to $\mathbf{Ax} = 0$, and is $\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$. The kernel has dimension 2.

2. $\mathbf{R} = \begin{pmatrix} \sqrt{6} & \sqrt{6} & \sqrt{6} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 2\sqrt{3} \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$

3. a. Solve $\mathbf{R\,x} = \mathbf{Q}^{\mathrm{T}}\mathbf{b}$ for least squares solution $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

b. $\mathbf{P} = \mathbf{QQ}^{\mathrm{T}} = \frac{1}{4}\begin{pmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{pmatrix}$

4. $\mathbf{S} = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{A}^{8} = \begin{pmatrix} 1021 & -1020 \\ 765 & -764 \end{pmatrix}$