Name________________________________ 14 December 2000

Final Exam

Math 4305 Andrew

Instructions: 1. You may use your text by Strang and one sheet of notes. Calculators may also be used.
2. Show your work and explain your answers and reasoning.
3. Do any 6 of the 7 problems. Clearly indicate the problem you *do not want graded* on the table below.

1. (25) Find all values of the parameter a for which the following system of equations has a solution.

$$\begin{aligned} x + 3y + 3z &= 1 \\ x + y + 6z &= a \\ -x + y - 9z &= a \end{aligned}$$

2. (25) A matrix **A** is said to be *skew-symmetric* if $\mathbf{A}^T = -\mathbf{A}$. Exhibit a basis for the vector space of all n by n skew-symmetric matrices and calculate the dimension of this vector space.

3. (25) Let P denote the plane spanned by the vectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$.

   a. Determine the matrix **R** for the orthogonal projection onto P.

   b. Determine the matrix **H** for the (orthogonal) reflection across P.

4. (25) Calculate the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 2 & 2 \\ -1 & 0 & 3 \end{pmatrix}$. Is **A** similar to a diagonal matrix?

5. (25) On the first hour test we saw that for every n by n matrix **A**, there is a polynomial $q(x)$ of degree at most n^2 such that $q(\mathbf{A}) = \mathbf{0}$ (the identically 0 matrix). We later proved the Cayley-Hamilton Theorem, which states that if **A** is any n by n matrix, and p denotes its characteristic polynomial, then $p(\mathbf{A}) = \mathbf{0}$. The characteristic polynomial, or course, has degree n.

The *minimal polynomial* of **A** is defined to be the polynomial m with leading coefficient 1 of smallest degree for which $m(\mathbf{A}) = \mathbf{0}$.

a. Calculate the characteristic and minimal polynomials of $\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

b. Prove that if **A** and **B** are similar matrices, then **A** and **B** have the same minimal polynomial.

6. (25) The *condition number* of a matrix **A** is defined to be $c(\mathbf{A}) = \|\mathbf{A}\| \|\mathbf{A}^{-1}\|$, and it provides a measure of the sensitivity (inherent and due to round-off error) of $\mathbf{A}\,\mathbf{x} = \mathbf{b}$. Although the condition number is usually only estimated, in this problem we'll ask you to actually calculate two of them.

a. Using $\|\mathbf{C}\|_2$ for all matrices, show that the condition number of any orthogonal matrix is 1.

b. The n by n *Hilbert matrix* arises from the normal equations for least-squares polynomial approximation, and is given by $\mathbf{H} = \left(h_{i,j}\right)$ with

$$h_{i,j} = \frac{1}{i+j-1}, \quad 1 \le i,j \le n.$$

Write down the 4 by 4 Hilbert matrix, and compute its condition number, using $\|\mathbf{C}\|_\infty$ for all matrices. The inverse of this matrix is

$$\begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix}.$$

7. (25) Suppose $\mathbf{A} = \left(a_{i,j}\right)$ is a matrix with $a_{i,j} > 0$ for all *i*,*j*, and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ is a nonzero vector with $v_j \geq 0$ for all *j*. Show that all coordinates of **A v** are strictly positive.

Answers

1. The system has a solution if and only if $a = 1/3$.

2. Let $\mathbf{B}_{i\,j}$ be the *n* by *n* matrix with a 1 in position *ij* and –1 in position *ji*. Then the collection of matrices $\mathbf{B}_{i\,j}$ with *i* < *j* is a basis for the space of skew-symmetric matrices. Thus the dimension is $\frac{n(n-1)}{2}$.

3. $\mathbf{R} = \frac{1}{17}\begin{pmatrix} 13 & 6 & -4 \\ 6 & 8 & 6 \\ -4 & 6 & 13 \end{pmatrix}$, $\mathbf{H} = 2\mathbf{P} - \mathbf{I}$.

4. $\lambda = 2, 2, 2$. Two linearly independent eigenvectors are $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. The matrix is not diagonalizable.

5. a. The characteristic polynomial is $(\lambda - 2)^3$. The minimal polynomial is $(\lambda - 2)^2$.

   b. Start by showing that if $\mathbf{A} = \mathbf{S}^{-1}\mathbf{B}\mathbf{S}$, then $p(\mathbf{A}) = \mathbf{S}^{-1}p(\mathbf{B})\mathbf{S}$.

6. a. If **Q** is orthogonal, then $\|\mathbf{Q}\|_2 = \sqrt{\text{max eigenvalue of } \mathbf{Q}^\mathbf{T}\mathbf{Q}} = 1$ since $\mathbf{Q}^\mathbf{T}\mathbf{Q} = \mathbf{I}$, and $\mathbf{Q}^{-1}$ is also orthogonal.

   b. Use the fact that $\|\mathbf{C}\|_\infty$ is the maximum absolute row sum of **C**.

7. Please see your class notes.