Math 4305 Summer 2003
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Here are two sample final exams.

Final Exam Fall 2000

1. (25) Find all values of the parameter a for which the following system of equations has a solution.

$$\begin{aligned} x + 3y + 3z &= 1 \\ x + y + 6z &= a \\ -x + y - 9z &= a \end{aligned}$$

2. (25) A matrix **A** is said to be *skew-symmetric* if $\mathbf{A}^{\mathbf{T}} = -\mathbf{A}$. Exhibit a basis for the vector space of all n by n skew-symmetric matrices and calculate the dimension of this vector space.

3. (25) Let P denote the plane spanned by the vectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$.

   a. Determine the matrix **R** for the orthogonal projection onto P.

   b. Determine the matrix **H** for the (orthogonal) reflection across P.

4. (25) Calculate the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 2 & 2 \\ -1 & 0 & 3 \end{pmatrix}$. Is **A** similar to a diagonal matrix?

5. (25) On the first hour test we saw that for every n by n matrix **A**, there is a polynomial $q(x)$ of degree at most n^2 such that $q(\mathbf{A}) = \mathbf{0}$ (the identically 0 matrix). We later proved the Cayley-Hamilton Theorem, which states that if **A** is any n by n matrix, and p denotes its characteristic polynomial, then $p(\mathbf{A}) = \mathbf{0}$. The characteristic polynomial, or course, has degree n.

The *minimal polynomial* of **A** is defined to be the polynomial m with leading coefficient 1 of smallest degree for which $m(\mathbf{A}) = \mathbf{0}$.

a. Calculate the characteristic and minimal polynomials of $\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

b. Prove that if **A** and **B** are similar matrices, then **A** and **B** have the same minimal polynomial.

6. (25) The *condition number* of a matrix **A** is defined to be $c(\mathbf{A}) = \|\mathbf{A}\|\,\|\mathbf{A}^{-1}\|$, and it provides a measure of the sensitivity (inherent and due to round-off error) of $\mathbf{A}\,\mathbf{x} = \mathbf{b}$. Although the condition number is usually only estimated, in this problem we'll ask you to actually calculate two of them.

a. Using $\|\mathbf{C}\|_2$ for all matrices, show that the condition number of any orthogonal matrix is 1.

b. The n by n *Hilbert matrix* arises from the normal equations for least-squares polynomial approximation, and is given by $\mathbf{H} = \left(h_{i,j}\right)$ with

$$h_{i,j} = \frac{1}{i+j-1}, \quad 1 \le i,j \le n.$$

Write down the 4 by 4 Hilbert matrix, and compute its condition number, using $\|\mathbf{C}\|_\infty$ for all matrices. The inverse of this matrix is

$$\begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix}.$$

7. (25) Suppose $\mathbf{A} = \left(a_{i,j}\right)$ is a matrix with $a_{i,j} > 0$ for all i,j, and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ is a nonzero vector with $v_j \ge 0$ for all j. Show that all coordinates of $\mathbf{A}\,\mathbf{v}$ are strictly positive.

Final Exam Summer 2001

1. For the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$

   a. Find the **LU** factorization for **A**.

   b. Use the **LU** factorization to solve $\mathbf{A}\ \mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

2. Determine

   a. A basis for the column space, and

   b. A basis for the nullspace

   of the matrix $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 & 4 \\ -1 & 2 & 1 & -1 & -2 \\ 1 & 1 & 2 & 2 & 3 \end{pmatrix}$.

3. Calculate the least squares solution of $\begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} x = \begin{pmatrix} -4 \\ -7 \\ 4 \\ 6 \end{pmatrix}$.

4. Either find the matrices requested, or explain why no such matrices exists.

a. A matrix **S** and a real diagonal matrix **D** such that $\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S} = \mathbf{D}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

b. An orthogonal matrix **Q** and a real diagonal matrix **D** such that $\mathbf{Q}^{\mathbf{T}}\mathbf{A}\mathbf{Q} = \mathbf{D}$, again for

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

c. A matrix **S** and a real diagonal matrix **D** such that $\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S} = \mathbf{D}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

5. a. Prove that if **U** and **V** are unitary, then $\mathbf{A} = \mathbf{U}^{\mathbf{H}}\mathbf{V}\,\mathbf{U}$ is also unitary.

b. Prove that if a matrix is both upper triangular and unitary, then it is diagonal.

c. Use your results from parts *a* and *b*, together with Schur's Lemma (see the index) to prove that every unitary matrix can be diagonalized by a unitary matrix. I.e., show that if **A** is unitary, then there exists a unitary matrix **U** such that $\mathbf{U}^{\mathbf{H}}\mathbf{A}\,\mathbf{U} = \mathbf{D}$ is diagonal.

d. Can the matrix **D** in part *c* be chosen to have real entries?

6. The function

$$f(x,y,z) = 2x^2 - 6x + 2xy - 8y + 3y^2 + 3zy - 3z + xyz - xz + 7$$

has a critical posint at (1,1,0). Determine whether the function has a local maximum, local minimum, or saddle point at (1,1,0).

7. For the linear system

$$\begin{aligned} 2x - y + z &= -1 \\ 2x + 2y + 2z &= 4 \\ -x - y + 2z &= -5 \end{aligned}$$

a. Find the Jacobi matrix for this system and compute its spectral radius.

b. Find the Gauss-Seidel matrix for this system and compute its spectral radius.

c. What conclusions can you draw from parts *a* and *b*?