Name________________________________ 25 September 2000

Math 4305 Andrew Hour Test I

Instructions:

1. Please do all 4 problems. Be sure to explain your work.
2. Closed book, calculators may be used.

1. Let $A = \begin{pmatrix} 1 & 2 & 0 & -3 & 1 \\ 2 & 1 & 3 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 \end{pmatrix}$.

   a. Find all solutions to $\mathbf{Ax} = \mathbf{0}$.

   b. Find bases for the nullspace of **A** and the column space of **A**.

2. Let A be an n x n matrix. Prove that the set $\left\{ I,\ A,\ A^2,\ .\ .\ .\ ,\ A^{n^2} \right\}$ is linearly dependent, and deduce that there exists a polynomial
$q(\lambda) = a_0 + a_1 \lambda + . . . + a_{n^2} \lambda^{n^2}$ of degree n^2 such that
$q(A) = a_0 I + a_1 A + . . . + a_{n^2} A^{n^2}$ is the zero matrix.

3. Find matrices for the linear transformations

   a. R = rotation of $\mathbf{R}^3$ about the z-axis by 90 degrees, clockwise as viewed from the positive z-axis.

   b. S = reflection of $\mathbf{R}^3$ across the y-z plane.

   c. P = orthogonal projection of $\mathbf{R}^3$ onto the x-z plane.

   d. R followed by S followed by P

4. a. (By applying the Gram-Schmidt process to the columns of A) find an orthogonal matrix Q and an upper triangular matrix R such that A = QR, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

   b. Use your result from part a to find the least squares solution to the system of equations

$$\begin{aligned} a \quad\quad\quad\quad &= 4 \\ a + b + c &= 0 \\ a - b + c &= 12 \\ a + 2b + 4c &= 20 \end{aligned}$$

Answers.

1. a. $\begin{pmatrix} -s - 2t \\ t + 2s \\ t \\ s \\ 0 \end{pmatrix}$ b. Basis for nullspace $\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. Basis for column space $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

2. Hint: What is the dimension of the space of n by n matrices?

3. a. $R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ b. $S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ c. $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

   d. $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4. a. $Q = \frac{1}{2} \begin{pmatrix} 1 & \frac{-1}{\sqrt{5}} & -1 \\ 1 & \frac{1}{\sqrt{5}} & -1 \\ 1 & \frac{-3}{\sqrt{5}} & 1 \\ 1 & \frac{-3}{\sqrt{5}} & 1 \end{pmatrix}$ $R = \begin{pmatrix} 2 & 1 & 3 \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & 2 \end{pmatrix}$ b. $\begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix}$