Name________________________________ 30 October 2000

Math 4305 Andrew Hour Test 2

Instructions:

1. Please do problem 1 and any 3 of problems 2 through 5. Indicate the problem you **do not want graded** in the box at right.
2. Closed book, calculators may be used.
3. Be sure to explain your work.

Do not grade

1. (40) (You must do problem 1.) For each matrix $\mathbf{A}$, determine

   a. If there exists a nonsingular matrix $\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{D}$ is diagonal, and

   b. If there exists an orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q}^{\mathrm{T}}\mathbf{AQ} = \mathbf{D}$ is diagonal.
   (Recall that a matrix $\mathbf{Q}$ is *orthogonal* if $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{QQ}^{\mathrm{T}} = \mathbf{I}$)

   Be sure to explain your reasoning.

   a. $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$ b. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

   c. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ d. $A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$

2. (20) Suppose $x_0 = 7$, $x_1 = 21$, and $x_{n+2} = \frac{3}{4}x_n + \frac{1}{4}x_{n+1}$.

   a. Verify that this difference equation may be written in the form

   $$\mathbf{u}_n = \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}, \quad \mathbf{u}_0 = \begin{pmatrix} 21 \\ 7 \end{pmatrix}, \quad \mathbf{u}_{n+1} = \begin{pmatrix} 1/4 & 3/4 \\ 1 & 0 \end{pmatrix}\mathbf{u}_n .$$

   b. Use this matrix formulation to calculate $\lim_{n\to\infty} x_n$.

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3. (20) A square matrix $\mathbf{A}$ with real entries is said to be *normal* if $\mathbf{A}^{\mathbf{T}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\mathbf{T}}$. (For complex entries, we use the hermitian transpose).

   a. Show that if $\mathbf{A}$ is a normal matrix with real entries, then for every $\mathbf{x}$, $\|\mathbf{A}\mathbf{x}\| = \|\mathbf{A}^{\mathbf{T}}\mathbf{x}\|$. (Hint: For any vector $\mathbf{w}$, $\|\mathbf{w}\|^2 = \langle \mathbf{w},\mathbf{w} \rangle$. (In matrix terms, $\|\mathbf{w}\|^2 = \mathbf{w}^{\mathbf{T}}\mathbf{w}$ for a column vector $\mathbf{w}$.))

   b. Use the result of part *a* to prove that if $\mathbf{A}$ is a normal matrix with real entries, then the nullspaces of $\mathbf{A}$ and $\mathbf{A}^{\mathbf{T}}$ are the same.

   c. Give an example of a 2 by 2 matrix with real entries which is not normal. (Points decrease as number of non-zero entries in your example increase!)

4. (20) Determine matrices Σ and $\mathbf{Q}_2$ and the first column of $\mathbf{Q}_1$ in the singular value decomposition $\mathbf{A} = \mathbf{Q}_1 \Sigma \mathbf{Q}_2^{\mathbf{T}}$ for $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

5. (20) Use the Cayley-Hamilton Theorem to calculate the inverse of $\mathbf{B} = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}$.

Answers.

1. a. Eigenvalues are distinct, so $\mathbf{S}$ exists, $\mathbf{A}$ is not symmetric, so no such $\mathbf{Q}$ exists.

   b. Eigenspace for $\lambda = 2$ is two dimensional, so $\mathbf{S}$ exists, but no such $\mathbf{Q}$ does.

   c. No such $\mathbf{S}$ (and hence no $\mathbf{Q}$).

   d. $\mathbf{Q}$ exists since $\mathbf{A}$ is symmetric.

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3. a. $\|Ax\|^2 = \langle Ax,Ax \rangle = \langle A^T Ax,x \rangle = \langle AA^T x,x \rangle = \|A^T x\|^2$ c. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

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4. $\Sigma = \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad q_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

5. $B^{-1} = \frac{1}{2}\left(B^2 - 4B + 5I\right)$