D'Alembert Solution to wave equation

12 February 2007

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Math 4581

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Example 1

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In the first example we will assume the initial position is a "roof function" of height one,supported on the interval [-1,1] and the initial velocity is 0.

> $f := x \rightarrow piecewise(-1 < x \text{ and } x < 0, x+1, 0 < x \text{ and } x < 1, 1-x);$

$$f := x \rightarrow piecewise(-1 < x \textbf{ and } x < 0,\ x+1,\ 0 < x \textbf{ and } x < 1,\ 1-x) \tag{1.1}$$

> $u1 := (x, t) \rightarrow \left(\frac{1}{2}\right) \cdot (f(x+t) + f(x-t));$

$$u1 := (x, t) \rightarrow \frac{1}{2} f(x+t) + \frac{1}{2} f(x-t) \tag{1.2}$$

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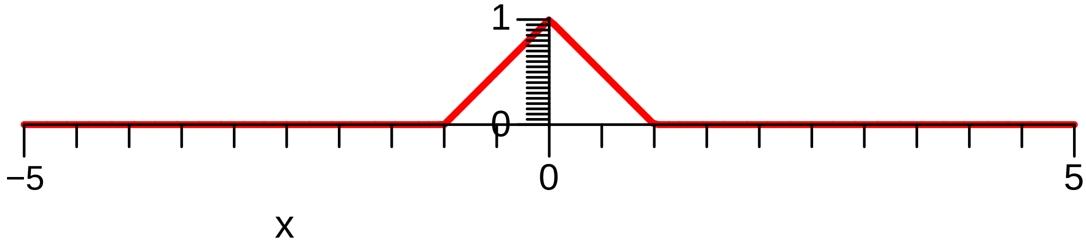
Here is a plot of the initial condition, which is of course the shape of the string when t = 0.

> $plot(f(x), x = -5..5, scaling = constrained, thickness = 2);$



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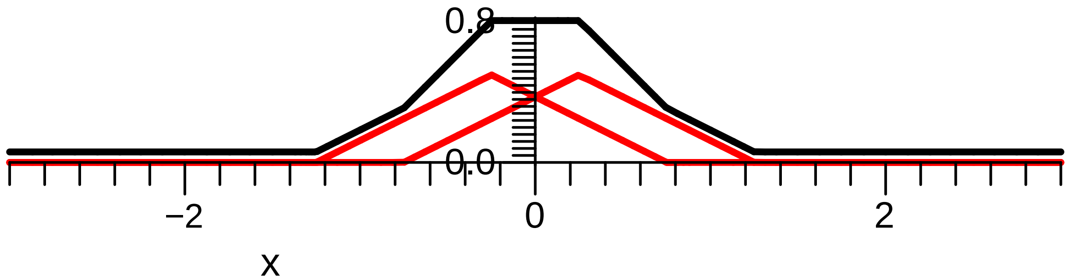
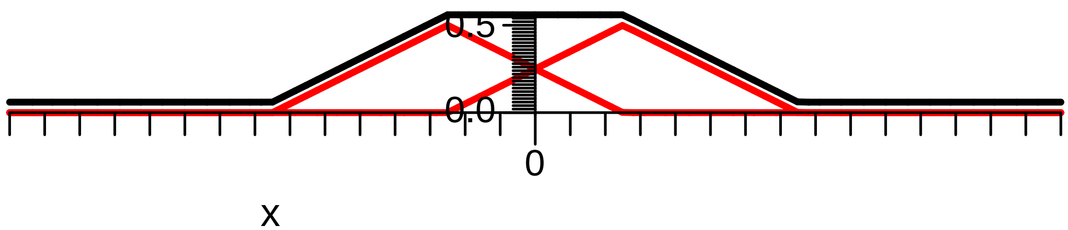
In the next few pictures we plot simultaneously 1/2 the translated initial conditions in red and their sum in black. I've actually moved the sum up a notch so that all graphs are visible.

> $plot\left(\left[u1(x, .25)+.06, \left(\frac{1}{2}\right)\cdot f(x+.25), \left(\frac{1}{2}\right)\cdot f(x-.25)\right], x=-3..3, scaling=constrained, color = [black, red, red], thickness = 2\right);$



> $plot([u1(x,.5) + .06, .5 \cdot f(x + .5), .5 \cdot f(x - .5)], x = -3..3, scaling = constrained, color = [black, red, red], thickness = 2);$



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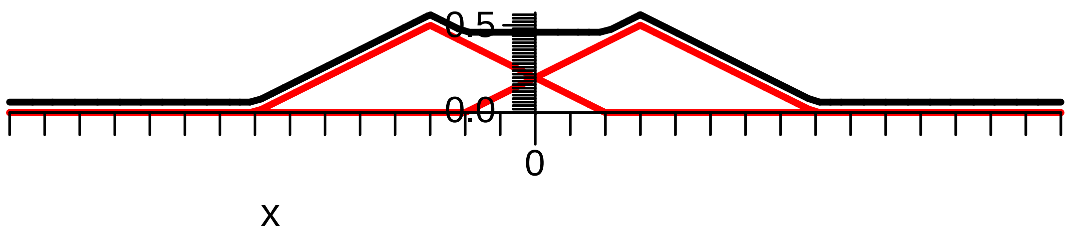
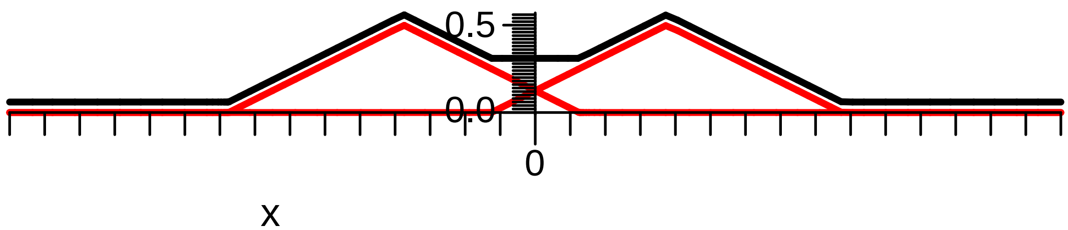
> $plot([u1(x,.6) + .06,.5 \cdot f(x + .6),.5 \cdot f(x-.6)], x = -3..3, scaling = constrained, color = [black, red, red], thickness = 2);$



> $plot\left(\left[u1(x,.75)+.06,.5\cdot f(x+.75),.5\cdot f(x-.75)\right], x=-3..3, scaling=constrained, color=[black, red, red], thickness=2\right);$



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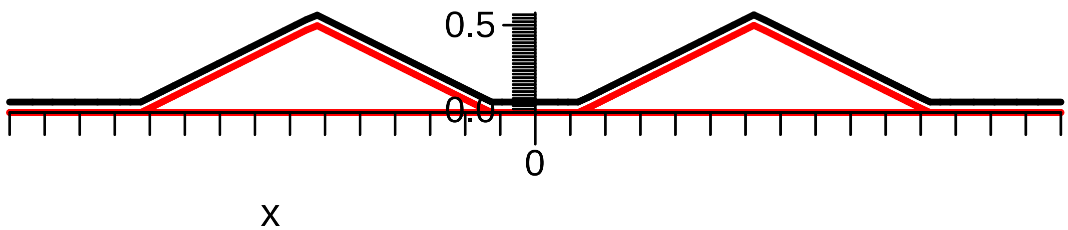
> *plot*([*u1*(*x*, 1.25) + .06,.5 · *f*(*x* + 1.25) ,.5·*f*(*x*−1.25)], *x* = −3 ..3, *scaling* = *constrained*, *color* = [*black*, *red*, *red*], *thickness* = 2);

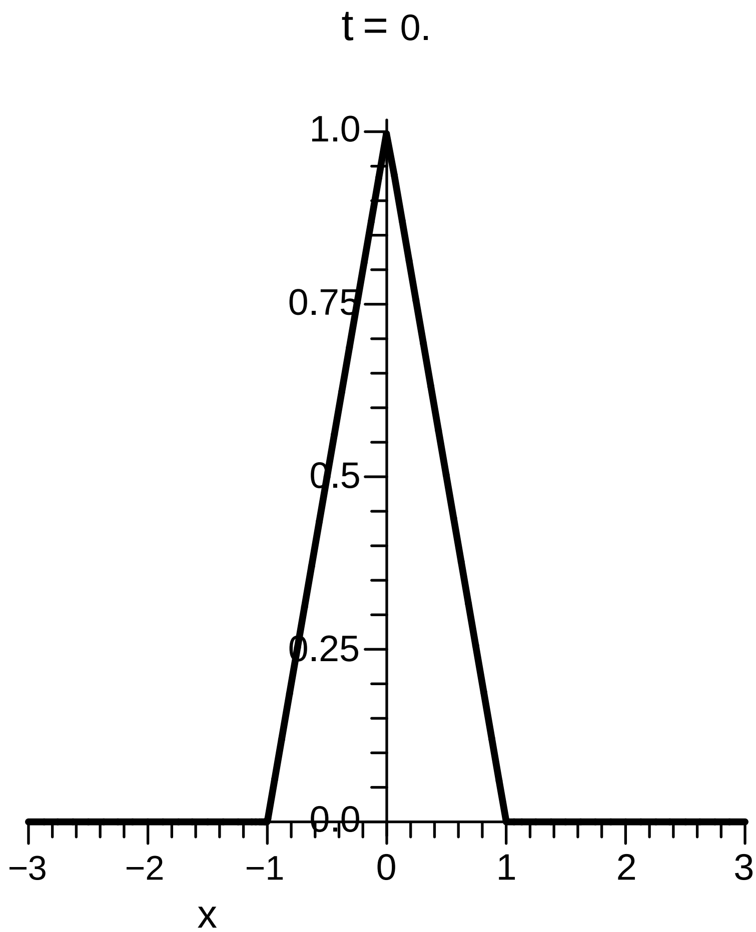


> $with(plots):$

> $v := x \rightarrow \left(\frac{1}{2}\right) \cdot (f(x+t) + f(x-t));$

$$v := x \rightarrow \frac{1}{2} f(x+t) + \frac{1}{2} f(x-t) \tag{1.3}$$

> $animate(plot, [v(x), x = -3..3, color = black, thickness = 2], t = 0..2);$



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Example 2

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In the second example we will assume the initial position is 0, and the initial velocity is a uniform pulse in the interval [–1,1]. As before, we will show individual plots for various values of t, and then an animation.

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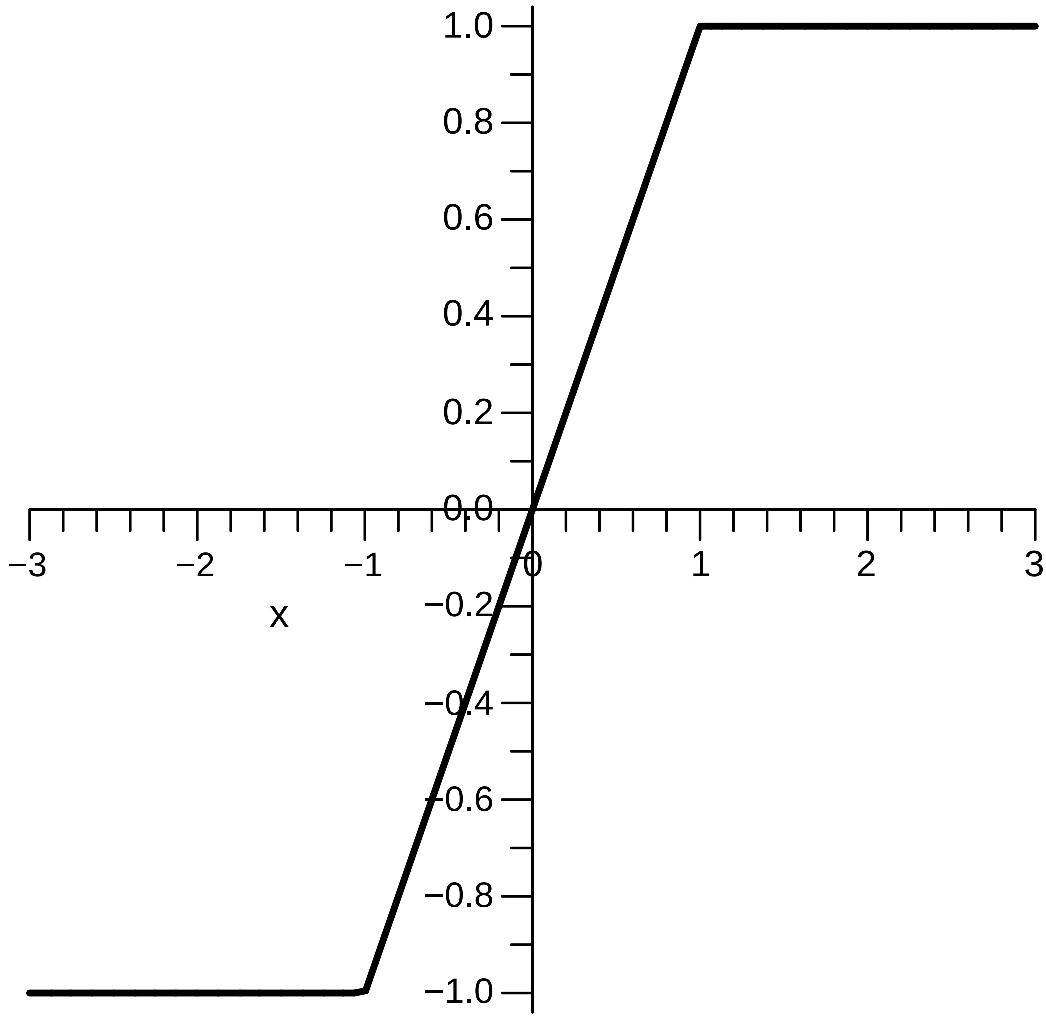
> $G := x \to piecewise(x < -1, -1, x > -1 \text{ and } x < 1, x, x > 1, 1);$

$$G := x \to piecewise(x < -1, -1, -1 < x \textbf{ and } x < 1, x, 1 < x, 1) \quad (2.1)$$

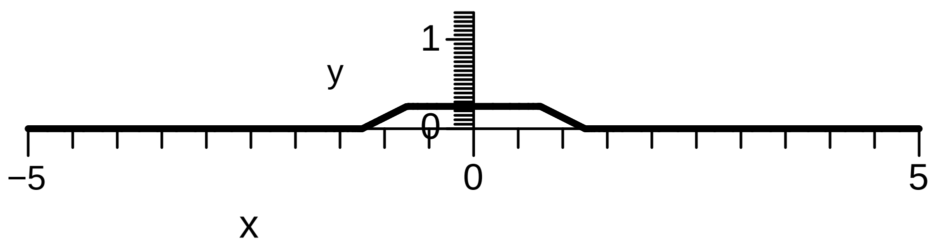
> $plot(G(x), x = -3..3, color = black, thickness = 2);$



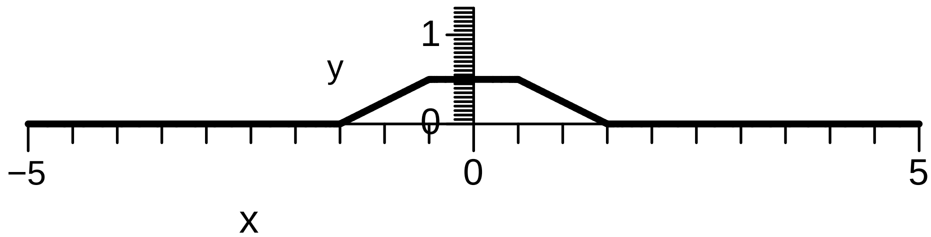
> $u2 := (x, t) \rightarrow \left(\frac{1}{2}\right) \cdot (G(x+t) - G(x-t));$

$$u2 := (x, t) \rightarrow \frac{1}{2}\, G(x+t) - \frac{1}{2}\, G(x-t) \tag{2.2}$$

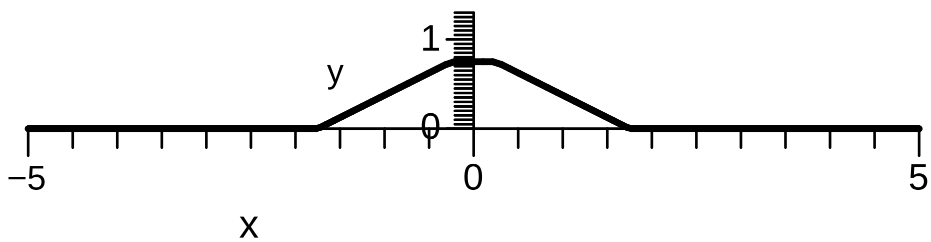
> $plot(u2(x, .25), x = -5..5, y = 0..1.3, scaling = constrained, color = black, thickness = 2);$



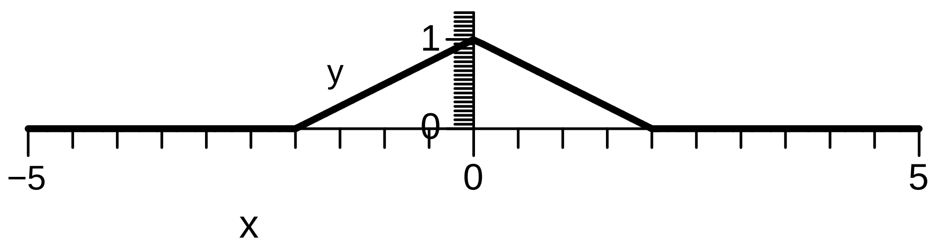
> $plot(u2(x, .5), x = -5..5, y = 0..1.3, scaling = constrained, color = black, thickness = 2);$



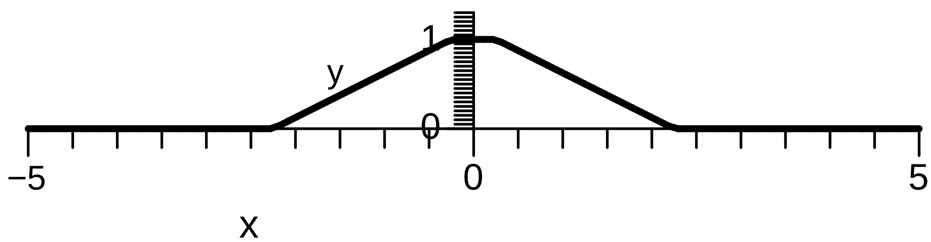
> $plot(u2(x, .75), x = -5..5, y = 0..1.3, scaling = constrained, color = black, thickness = 2);$



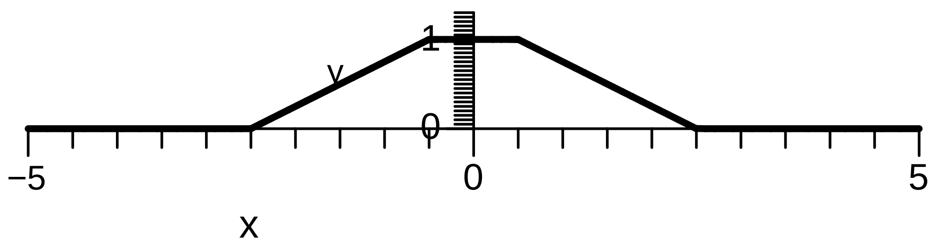
> *plot*(*u2*(*x*, 1), *x* = −5 ..5, *y* = 0 ..1.3, *scaling* = *constrained*, *color* = *black*, *thickness* = 2);



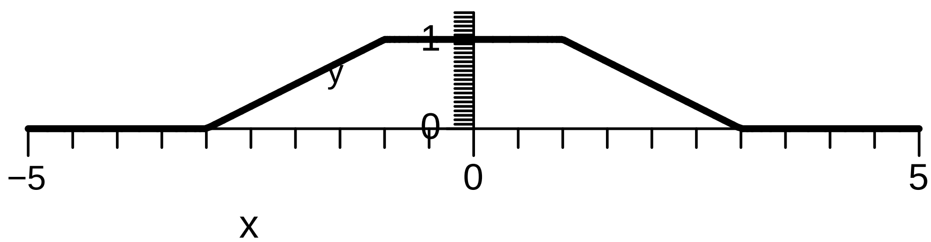
> $plot(u2(x, 1.25), x = -5..5, y = 0..1.3, scaling = constrained, color = black, thickness = 2);$



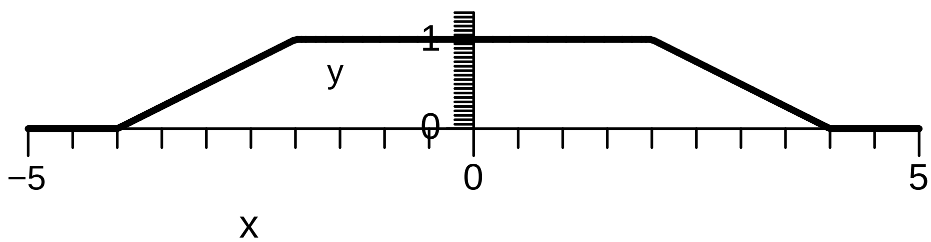
> $plot(u2(x, 1.5), x = -5..5, y = 0..1.3, scaling = constrained, color = black, thickness = 2);$



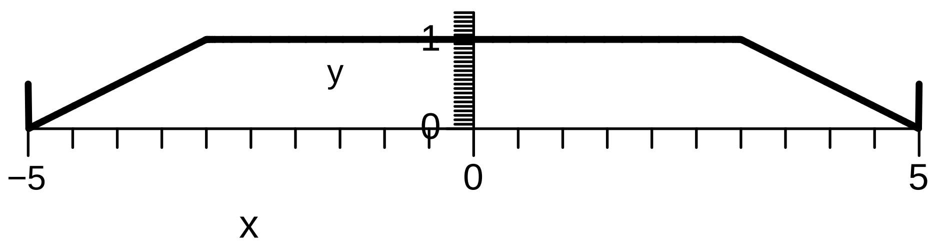
> $plot(u2(x, 2), x = -5..5, y = 0..1.3, scaling = constrained, color = black, thickness = 2);$



> *plot*(*u2*(*x*, 3), *x* = −5 ..5, *y* = 0 ..1.3, *scaling* = *constrained*, *color* = *black*, *thickness* = 2);



> $plot(u2(x, 4), x = -5..5, y = 0..1.3, scaling = constrained, color = black, thickness = 2);$



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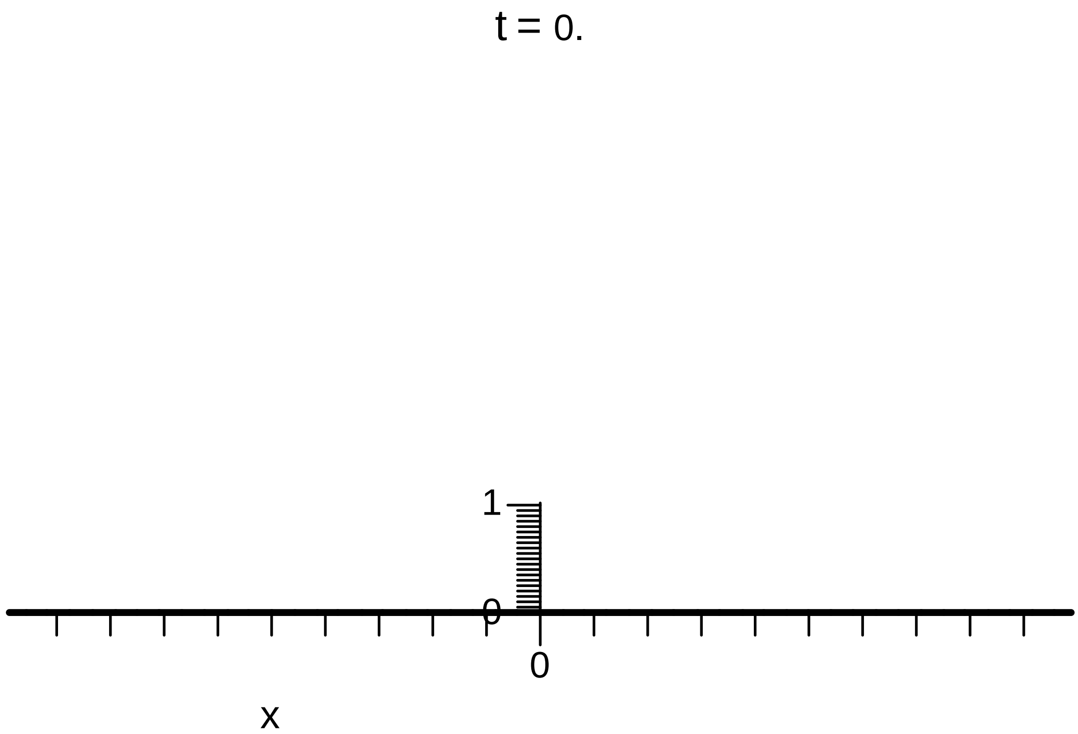
> *animate*(*plot*, [*u2*(*x*, *t*), *x* = −5 ..5, *color* = *black*, *thickness* = 2], *t* = 0 ..3, *scaling* = *constrained*);



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Example 3

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In this example we show two waves interacting.

> $f3a := x \rightarrow piecewise(-1 < x \text{ and } x < 0, x + 1, 0 < x \text{ and } x < 1, 1 - x);$

$$f3a := x \rightarrow piecewise(-1 < x \textbf{ and } x < 0, x + 1, 0 < x \textbf{ and } x < 1, 1 - x) \quad (2.1.1)$$

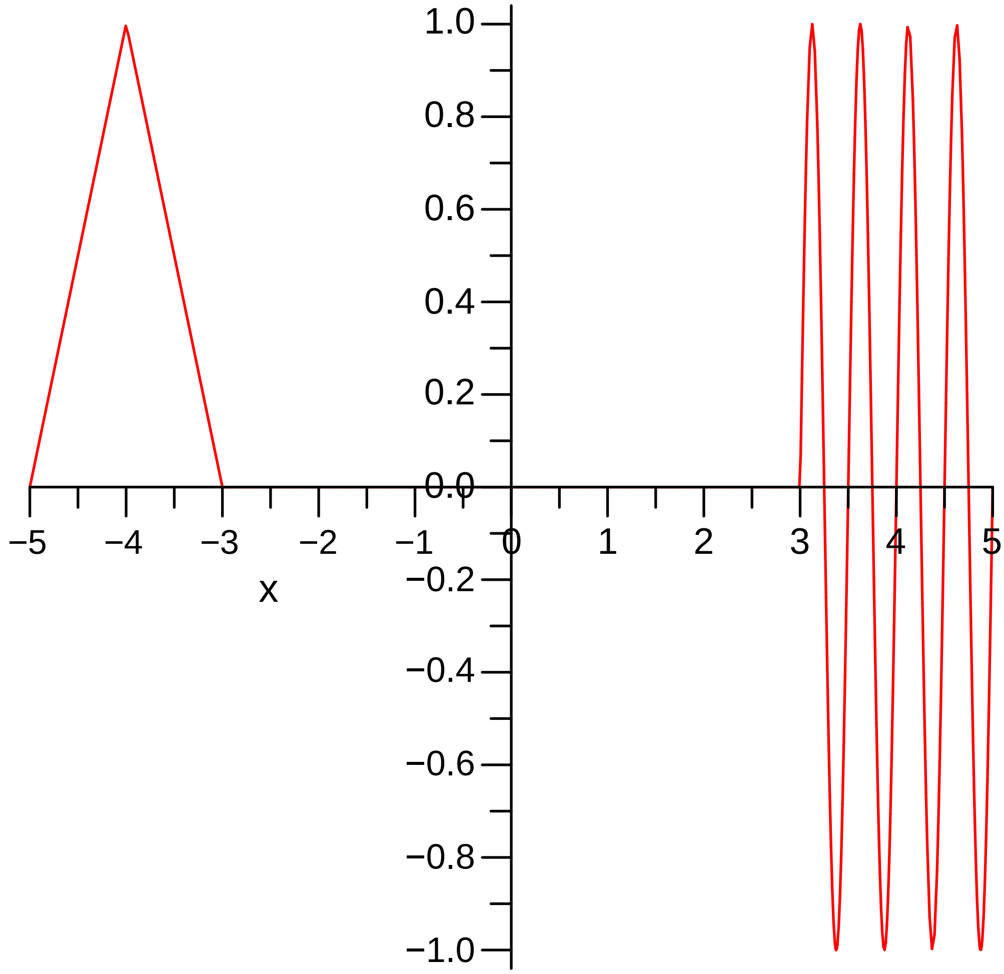
> $f3b := x \rightarrow piecewise(-1 < x \text{ and } x < 1, \sin(4 \cdot \pi \cdot x));$

$$f3b := x \rightarrow piecewise(-1 < x \textbf{ and } x < 1, \sin(4\,\pi\,x)) \quad (2.1.2)$$

> $f3 := x \rightarrow f3a(x + 4) + f3b(x - 4);$

$$f3 := x \rightarrow f3a(x + 4) + f3b(x - 4) \quad (2.1.3)$$

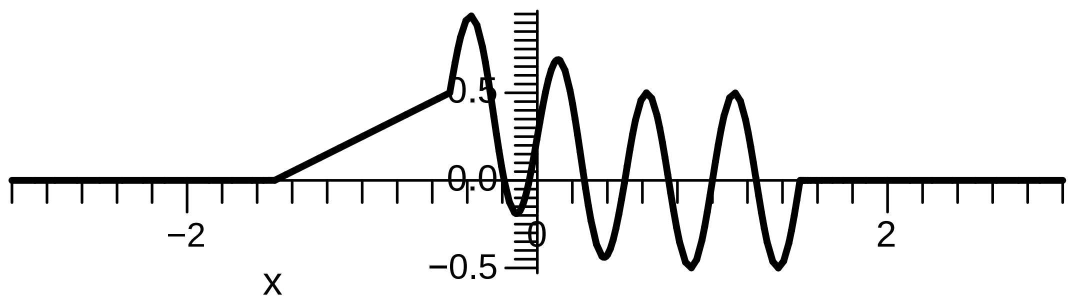
> $plot(f3(x), x = -5..5);$



> $u3 := (x, t) \rightarrow \left(\frac{1}{2}\right) \cdot (f3(x+t) + f3(x - t));$

$$u3 := (x, t) \rightarrow \frac{1}{2} f3(x+t) + \frac{1}{2} f3(x-t) \qquad (2.1.4)$$

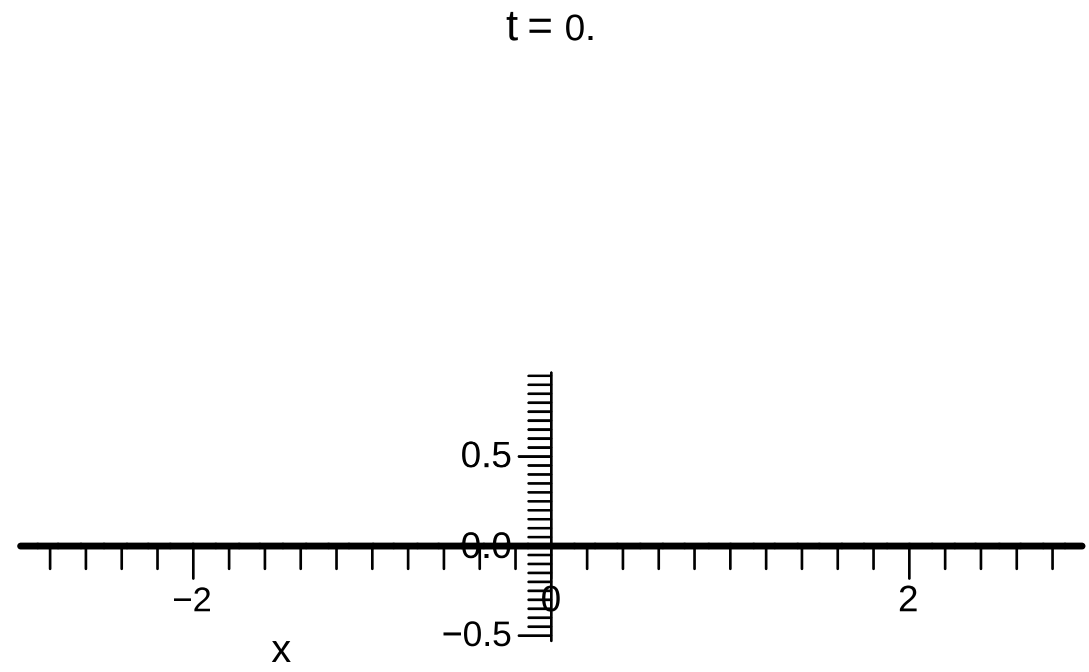
> *plot*($u3(x, 3.5)$, $x = -3..3$, *color* = *black*, *scaling* = *constrained*, *thickness* = 2);



> $with(plots):$

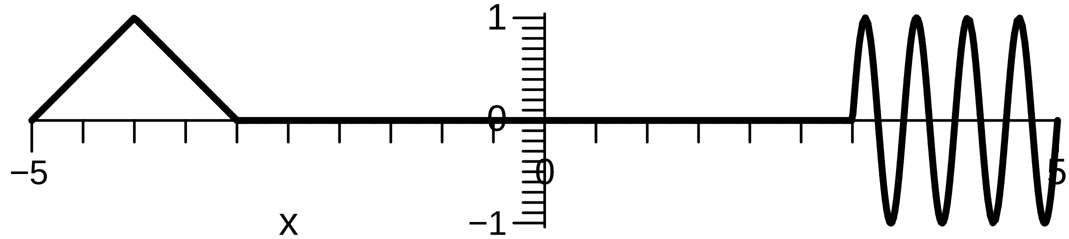
Warning, the name changecoords has been redefined

> $animate(plot, [u3(x, t), x = -3..3, color = black, thickness = 2], t = 0..6, scaling = constrained);$



> $animate(plot, [u3(x, t), x = -5..5, color = black, thickness = 2], t = 0..6, scaling = constrained);$

t = 0.



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