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Heat Equation Examples

25 January 2007

Math 4581

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Example 1

uxx = ut
u(0,t) = 10
u(1,t) = 100
u(x,0) = 0

The time independent solution to PDE and BC is

```
> v1 := x -> 10 + 90*x;
```

$$v1 := x \to 10 + 90\,x \tag{1.1}$$

So the initial conditions for the problem with homogeneous boundary conditions are

```
> g1 := x -> 0 - v1(x);
```

$$g1 := x \to -v1(x) \tag{1.2}$$

and these initial conditions have Fourier sine coefficients are

```
> a1 :=  n -> 2*int((-10 - 90*x)*sin(n*Pi*x),
           x = 0..1);
```

$$a1 := n \to 2 \int_0^1 (-10 - 90\,x)\sin(n\,\pi\,x)\,\mathrm{d}x \tag{1.3}$$

```
> a1(n);
```

$$\frac{20\,(-n\,\pi + 10\cos(n\,\pi)\,n\,\pi - 9\sin(n\,\pi))}{n^2\,\pi^2} \tag{1.4}$$

and the partial sums of the solution to the related problem with homogeneous BC are

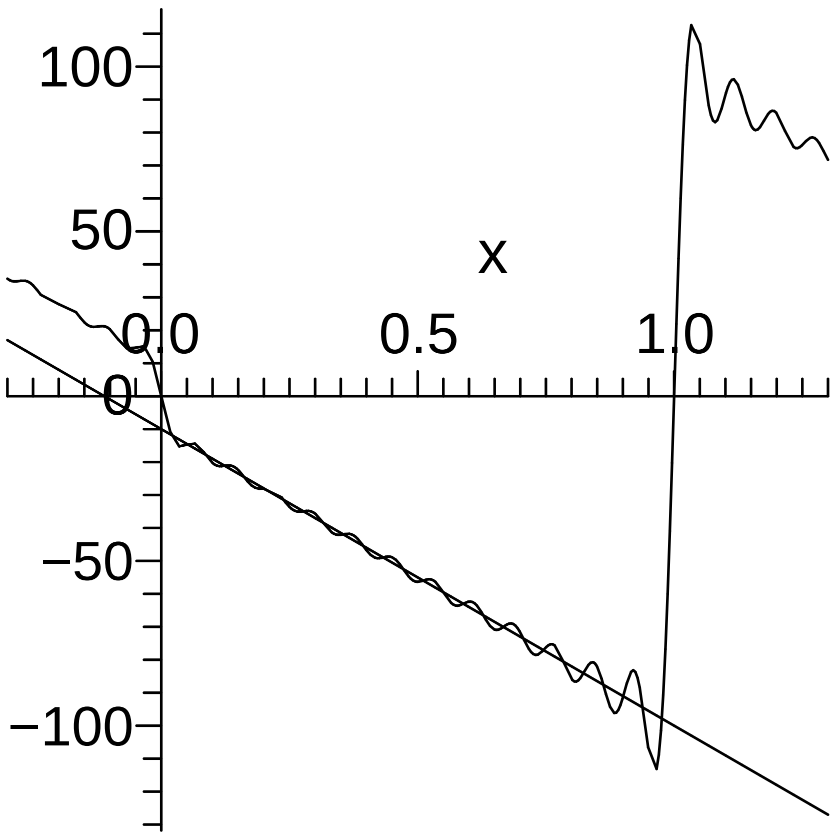
```
> w1N := (x,t,N) -> add(a1(n)*sin(n*Pi*x)*
          exp(-n^2*Pi^2*t),n=1..N);
```

$$w1N := (x, t, N) \to add\left(a1(n)\sin(n\,\pi\,x)\,\mathrm{e}^{(-n^2\,\pi^2\,t)},\ n = 1..N\right) \tag{1.5}$$

>

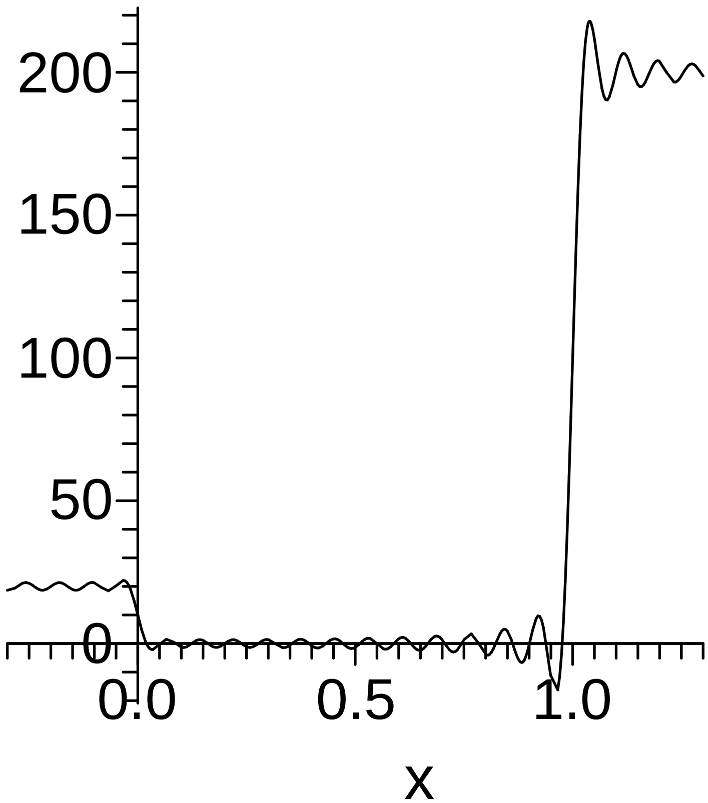
Here is a picture of one of these partial sums at t = 0 and the function – v1(x) = −10 −90x for which it is an approximation.

> *plot*([$-v1(x)$, $w1N(x, 0, 25)$], $x = -0.3..1.3$, *color* = *black*);



And another interesting plot

> *plot*([$-v1(x) + v1(x)$, $w1N(x, 0, 25) + v1(x)$], $x = -0.3..1.3$,
color = *black*);

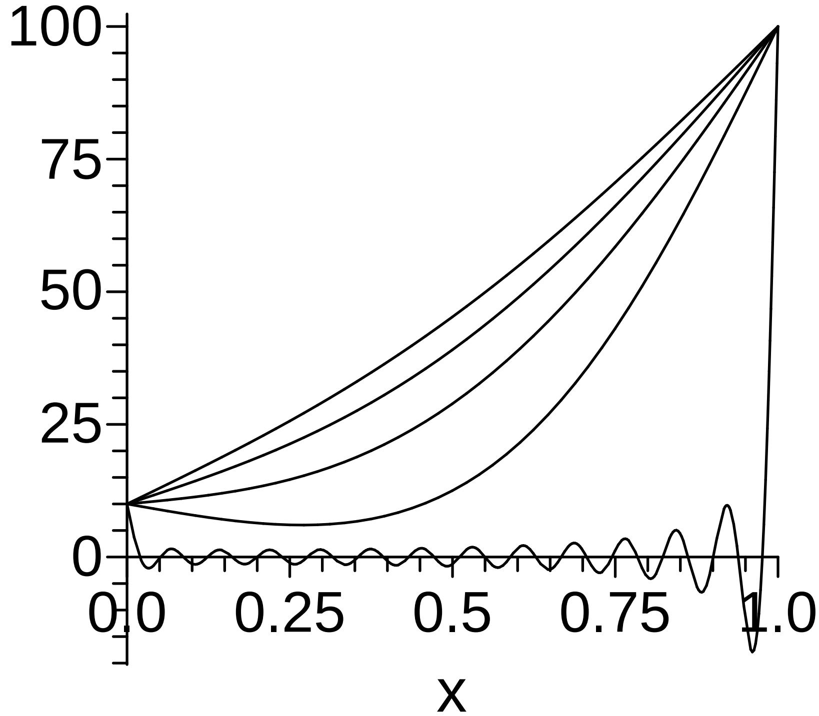


Finally, a partial sum to the solution of the original problem is

```
> u1 := (x,t,N) -> w1N(x,t,N) + v1(x);
```

$$u1 := (x, t, N) \rightarrow w1N(x, t, N) + v1(x) \tag{1.6}$$

```
> plot([seq(u1(x,.05*k,25),k=0..4)],
         x=0..1,color = black);
```

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>
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Example 2

uxx = ut
u(0,t) = 10
u(1,t) = 100
u(x,0) = 10 if 0 < x < .5, 100 if .5 < x < 1

The time independent solution is again

```
> v2 := x -> 10 + 90*x;
```

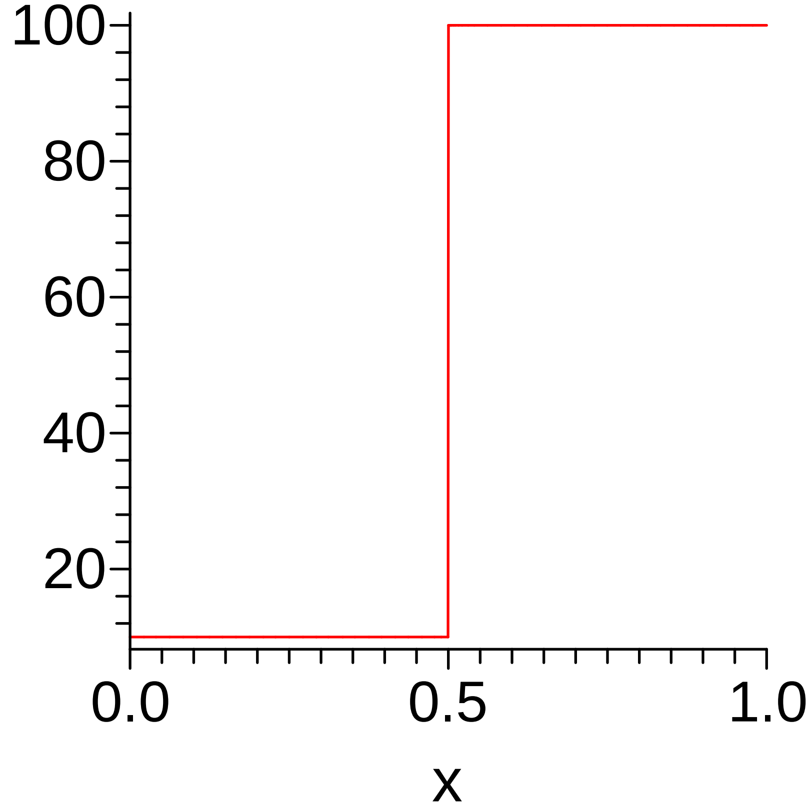
$$v2 := x \rightarrow 10 + 90\,x \qquad (2.1)$$

The original initial conditions are

```
> f2 := x -> piecewise(x<.5,10,100);
```

$$f2 := x \rightarrow piecewise(x < 0.5,\ 10,\ 100) \qquad (2.2)$$

```
> plot(f2(x),x=0..1);
```




```
>
```

So the initial conditions for the problem with homogeneous boundary conditions are

```
> g2 := x -> f2(x) - v2(x);
```

$$g2 := x \rightarrow f2(x) - v2(x) \qquad (2.3)$$

And the Fourier sine coefficients for these initial conditions are

```
> a2 :=  n -> 2*int(g2(x)*sin(n*Pi*x),
                x = 0..1);
```

$$a2 := n \rightarrow 2 \int_0^1 g2(x) \sin(n \pi x) \, dx \qquad (2.4)$$

```
> a2(n);
```

$$(9.118906528\ (-2.\ \sin(1.570796327\ n) \qquad (2.5)$$

$$+ 3.141592654 \cos(1.570796327\ n)\ n)) / n^2 + \frac{1}{n^2}\ (9.118906528\ ($$

$$2.\sin(1.570796327\,n) + 3.141592654\cos(1.570796327\,n)\,n - 2.\sin(3.141592654\,n)))$$

and the partial sums of the solution to the related problem with homogeneous BC are

```
> w2N := (x,t,N) -> add(a2(n)*sin(n*Pi*x)*
                   exp(-n^2*Pi^2*t),n=1..N);
```

$$w2N := (x, t, N) \rightarrow add\left(a2(n)\sin(n\pi x)\,\mathrm{e}^{(-n^2\pi^2 t)},\ n = 1..N\right) \quad (2.6)$$

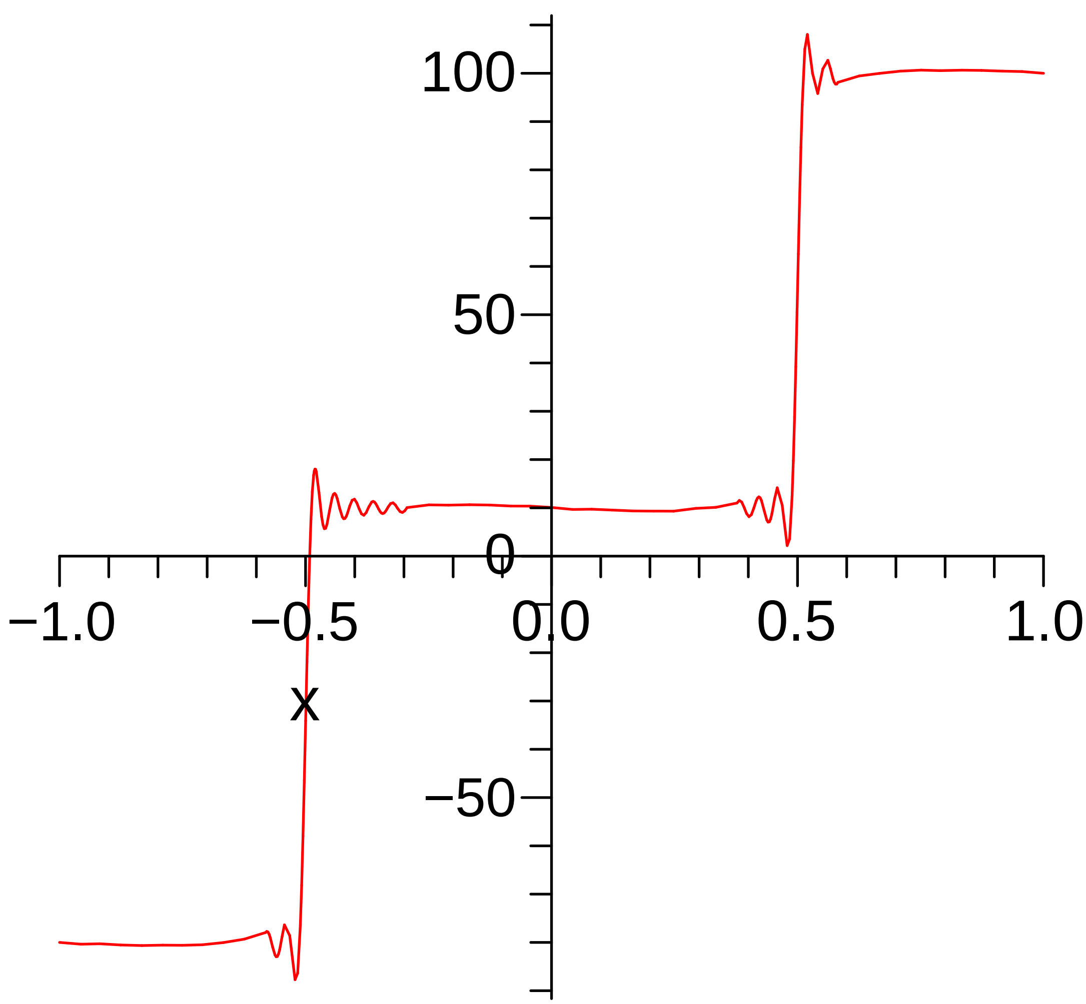
```
>
```

Finally, a partial sum to the solution of the original problem is

```
> u2 := (x,t,N) -> w2N(x,t,N) + v2(x);
```

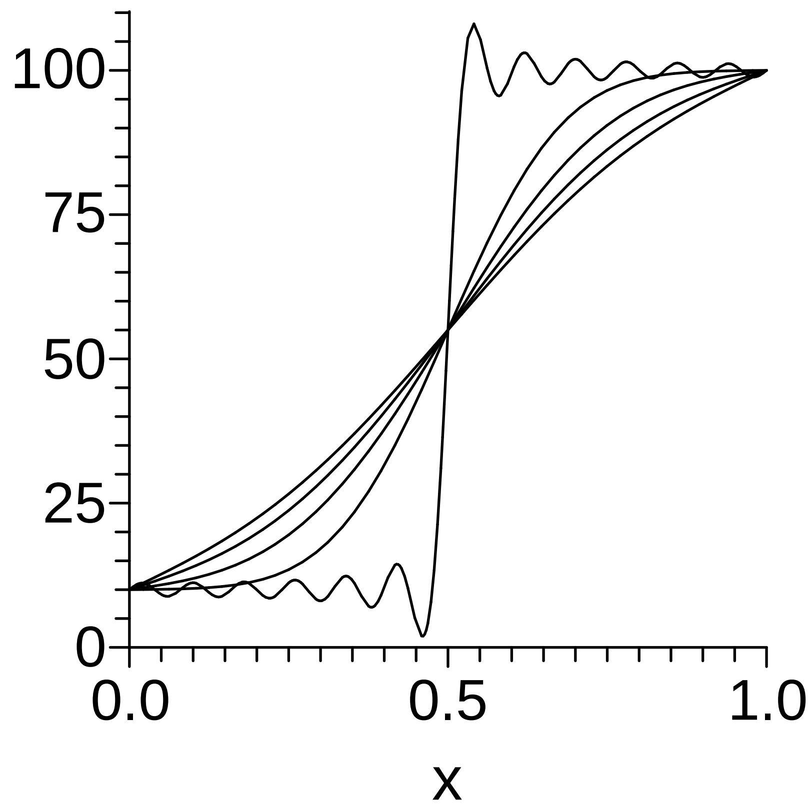
$$u2 := (x, t, N) \rightarrow w2N(x, t, N) + v2(x) \quad (2.7)$$

```
> plot(u2(x,0,50),x=-1..1);
```




```
> plot([seq(u2(x,.01*k,25),k=0..4)],
```

```
x=0..1,color = black);
```



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Example 3

uxx = ut
ux(0,t) = 0
u(1,t) = 100
u(x,0) = 100 x

The time independent solution to the PDE and BC is

> $v3 := x \to 100;$

$$v3 := x \to 100 \tag{3.1}$$

>

The original initial conditions are

> $f3 := x \rightarrow 100 \cdot x;$

$$f3 := x \rightarrow 100\, x \tag{3.2}$$

>

So the initial conditions for the problem with homogeneous boundary conditions are

> $g3 := x \rightarrow f3(x) - v3(x);$

$$g3 := x \rightarrow f3(x) - v3(x) \tag{3.3}$$

>

We will have a Fouier series involving cosine of odd multiples of (x Pi)/2

> $a3 := n \rightarrow 2 \cdot int\left(g3(x) \cdot cos\left(\frac{(2 \cdot n - 1)}{2} \cdot \pi \cdot x\right), x = 0..1\right);$

$$a3 := n \rightarrow 2 \int_0^1 g3(x) \cos\left(\left(n - \frac{1}{2}\right) \pi\, x\right) \mathrm{d}x \tag{3.4}$$

>

and the partial sums of the related problem with homogeneous boundary conditions are

> $w3N := (x, t, N) \rightarrow add\left(a3(n) \cdot cos\left(\frac{(2 \cdot n - 1)}{2} \cdot \pi \cdot x\right) \cdot exp\left(-\left(\frac{(2 \cdot n - 1)}{2} \cdot \pi\right)^2 \cdot t\right), n = 1..N\right);$

$$w3N := (x, t, N) \rightarrow add\left(a3(n) \cos\left(\left(n - \frac{1}{2}\right) \pi\, x\right) \mathrm{e}^{\left(-\left(n - \frac{1}{2}\right)^2 \pi^2 t\right)}, n = 1..N\right) \tag{3.5}$$

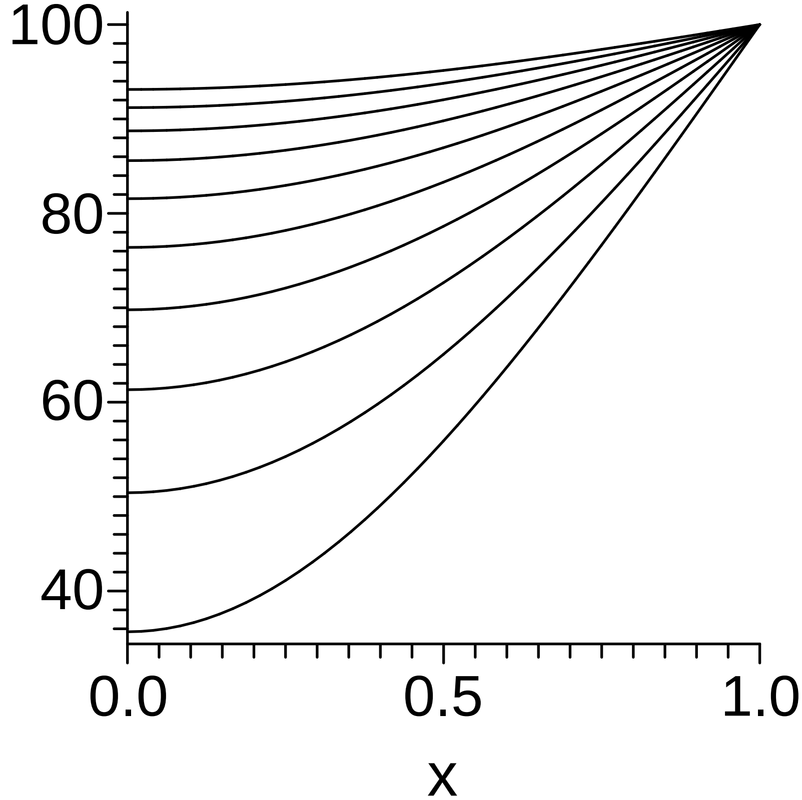
>

and a partial sum for the original problem is

> $u3 := (x, t, N) \rightarrow w3N(x, t, N) + v3(x);$

$$u3 := (x, t, N) \rightarrow w3N(x, t, N) + v3(x) \tag{3.6}$$

> $plot([seq(u3(x, 0.1 \cdot k, 25), k = 1..10)], x = 0..1, color = black);$



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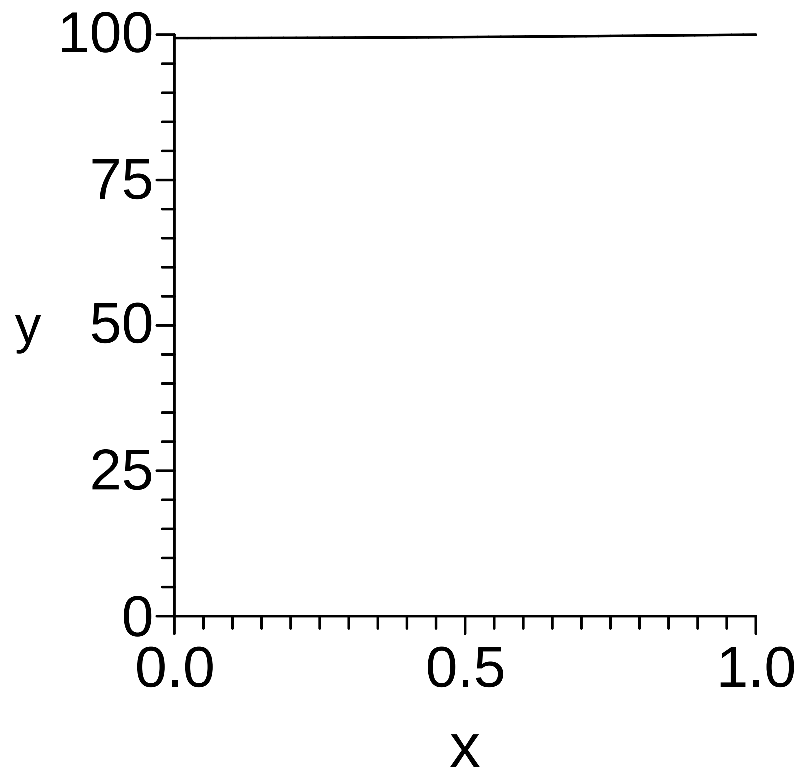
Here's a closer look for a large value of t.

> $plot(u3(x, 2, 25), x = 0..1, y = 0..100, color = black);$



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Example 4

This time we insulate both ends, and the boundary conditions are homegeneous to begin with. There is no need to subtract a steady state solution to achieve homegeneous boundary conditions, and indeed, there is not a unique time independent solution anyway.

The problem is

uxx = ut, 0 < x < 1, 0 < t
ux(0,t) = ux(1,t) = 0

u(x,0) a "roof" function 2x for 0 < x < .5 and 2 - 2x for .5 < x < 1.

As seen in class the solution will involve cos(nPix) n = 1... and a constant term a0.

The constant term is the average value uf the initial conditions, which is 1/2, and the other coefficients are

> $a4 := n \rightarrow 2\cdot\left(int\left(2\cdot x\cdot cos(n\cdot \pi\cdot x), x = 0..\frac{1}{2}\right) + int\left((2 - 2\cdot x)\cdot cos(n\cdot \pi\cdot x), x = \frac{1}{2}..1\right)\right);$

$$a4 := n \rightarrow 2\int_0^{\frac{1}{2}} 2\,x\cos(n\,\pi\,x)\,dx + 2\int_{\frac{1}{2}}^{1} (2-2\,x)\cos(n\,\pi\,x)\,dx \qquad (4.1)$$

>

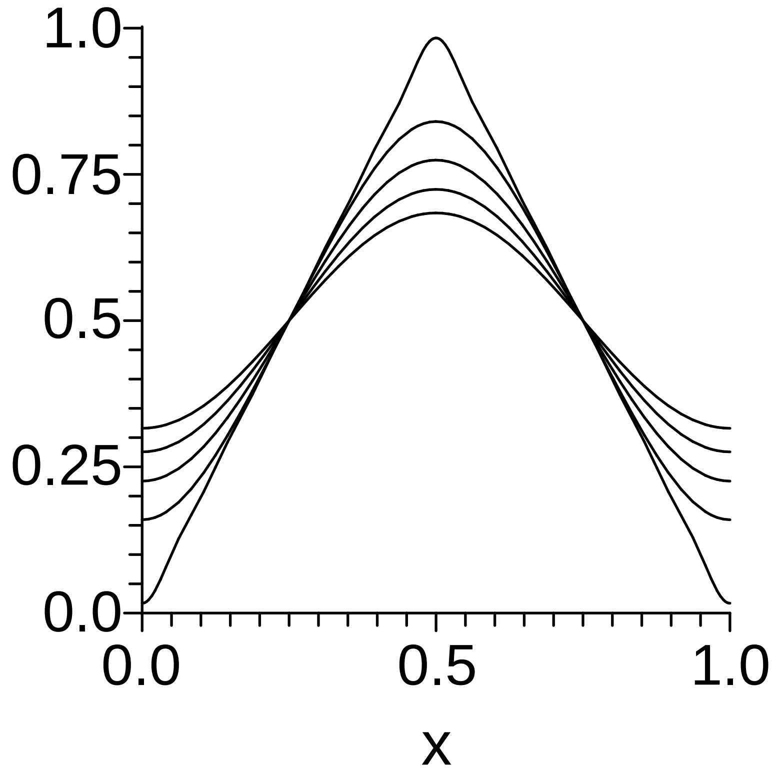
and our solution has partial sum

> $u4N := (x, t, N) \rightarrow \left(\frac{1}{2}\right) + add\left(a4(n)\cdot cos(n\cdot \pi\cdot x)\cdot exp\left(-n^2\cdot \pi^2\cdot t\right), n = 1..N\right);$

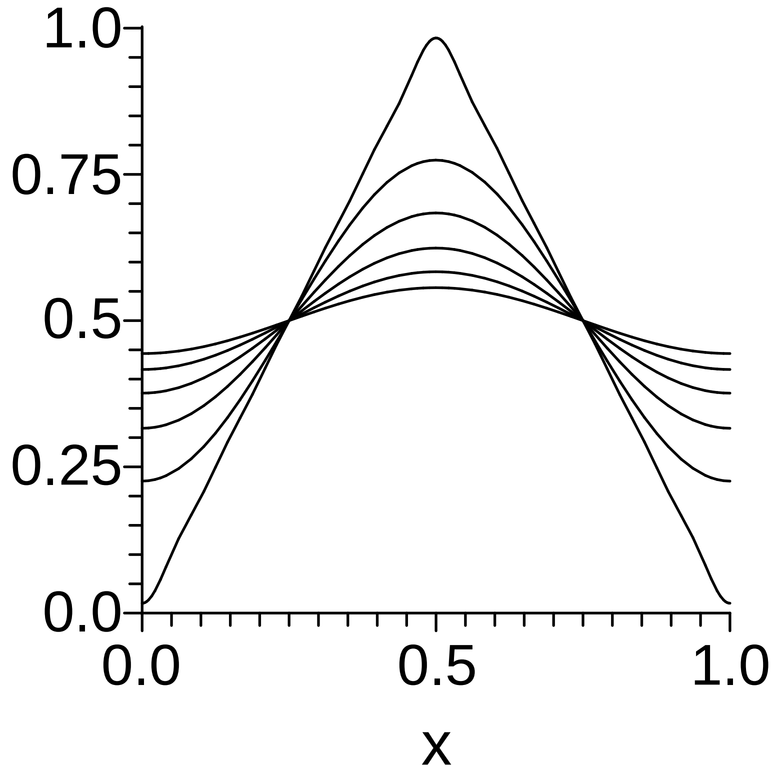
$$u4N := (x, t, N) \rightarrow \frac{1}{2} + add\left(a4(n)\cos(n\,\pi\,x)\,e^{(-n^2\,\pi^2\,t)}, n = 1..N\right) \qquad (4.2)$$

Here are three pictures.

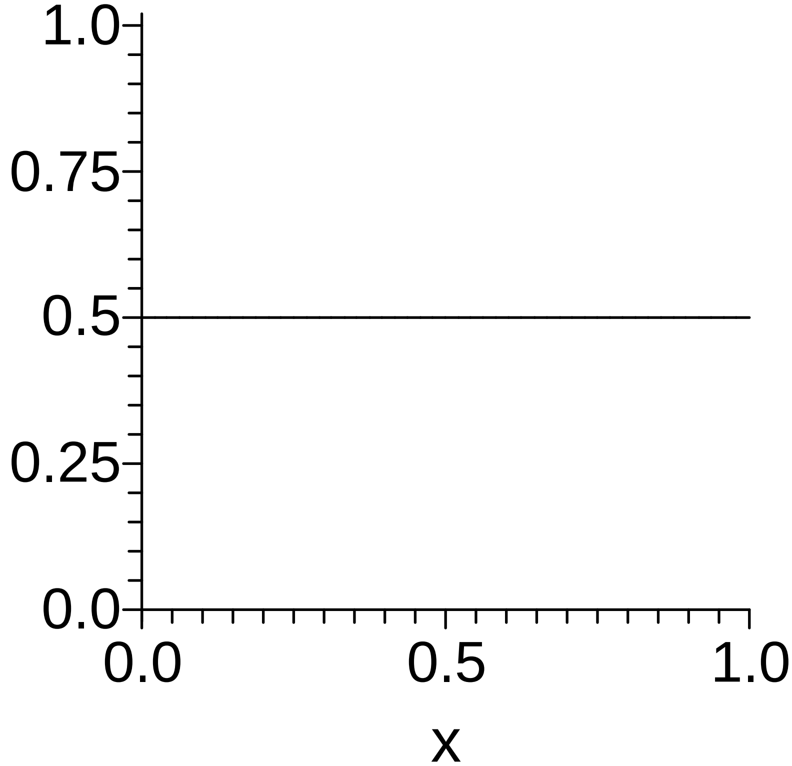
> $plot([seq(u4N(x, 0.005\cdot k, 25), k = 0..4)], x = 0..1, color = black);$



> $plot([seq(u4N(x, 0.01 \cdot k, 25), k = 0..5)], x = 0..1, color = black)$



> $plot(u4N(x, 1, 25), x = 0..1, color = black);$


1.0
0.75
0.5
0.25
0.0
0.0
0.5
1.0
x

>