Heat Equation Examples

Generation and Convection

30 January 2007

Math 5581

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Example 1 - HE with generation

Heat equation with generation.

uxx + 1 = ut 0 < x < 1, 0 < t
u(0,t) = 0
u(1,t) = 0
u(x,0) = 0

The time independent solution is

> $v := x \rightarrow -\frac{1}{2}\cdot x^2 + \frac{1}{2}\cdot x;$

$$v := x \rightarrow -\frac{1}{2}x^2 + \frac{1}{2}x \tag{1.1}$$

>

so we now let w = u - v(x), and w satsfies

wxx = wt
w(0,t) = w(1,t) = 0
w(x,0) = -v(x)

The Fourier coefficients are

> $a := n \rightarrow 2\cdot int\left(\left(\frac{1}{2}\right)\cdot\left(x^2 - x\right)\cdot sin(n\cdot \pi\cdot x), x = 0..1\right);$

$$a := n \rightarrow 2\int_0^1 \frac{1}{2}\left(x^2 - x\right)\sin(n\,\pi\,x)\,dx \tag{1.2}$$

> $a(n);$

$$\frac{-2 + n\,\pi\,\sin(n\,\pi) + 2\cos(n\,\pi)}{n^3\,\pi^3} \qquad (1.3)$$

>

and a partial sum for w is

> $w1N := (x, t, N) \rightarrow add\left(a(n)\cdot exp\left(-(n\cdot\pi)^2\cdot t\right)\cdot sin(n\cdot\pi\cdot x), n = 1..N\right);$

$$w1N := (x, t, N) \rightarrow add\left(a(n)\,e^{(-n^2\pi^2 t)}\sin(n\,\pi\,x), n = 1..N\right) \qquad (1.4)$$

>

and a partial sum for the solution to the original problem is

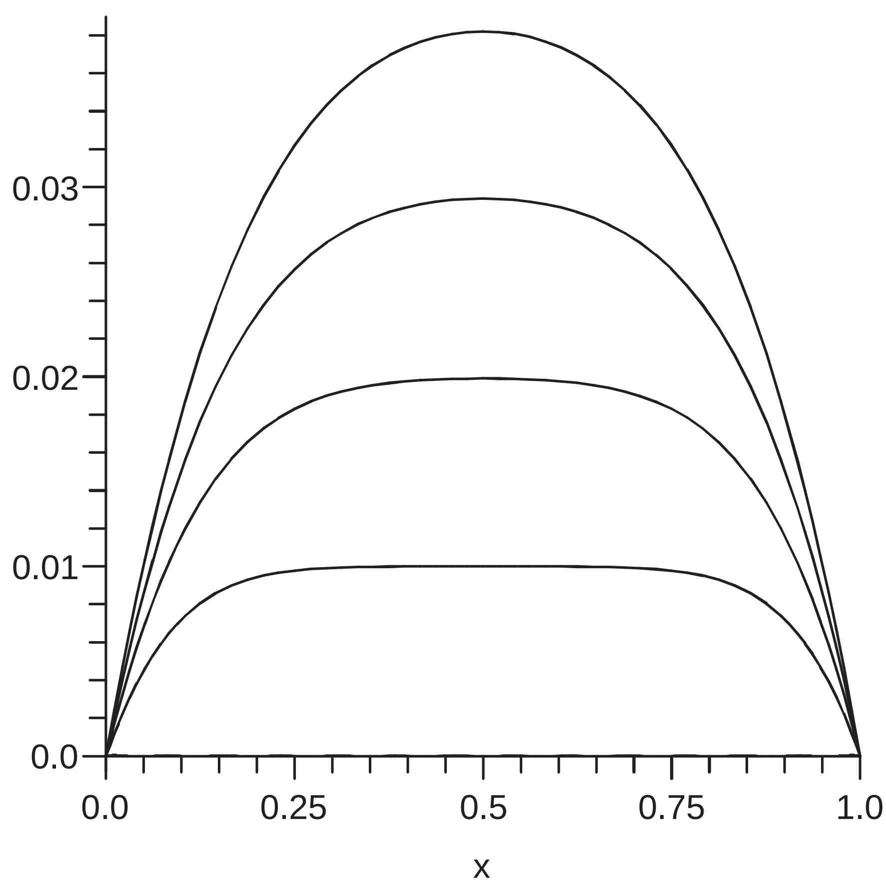
> $u1N := (x, t, N) \rightarrow v(x) + w1N(x, t, N);$

$$u1N := (x, t, N) \rightarrow v(x) + w1N(x, t, N) \qquad (1.5)$$

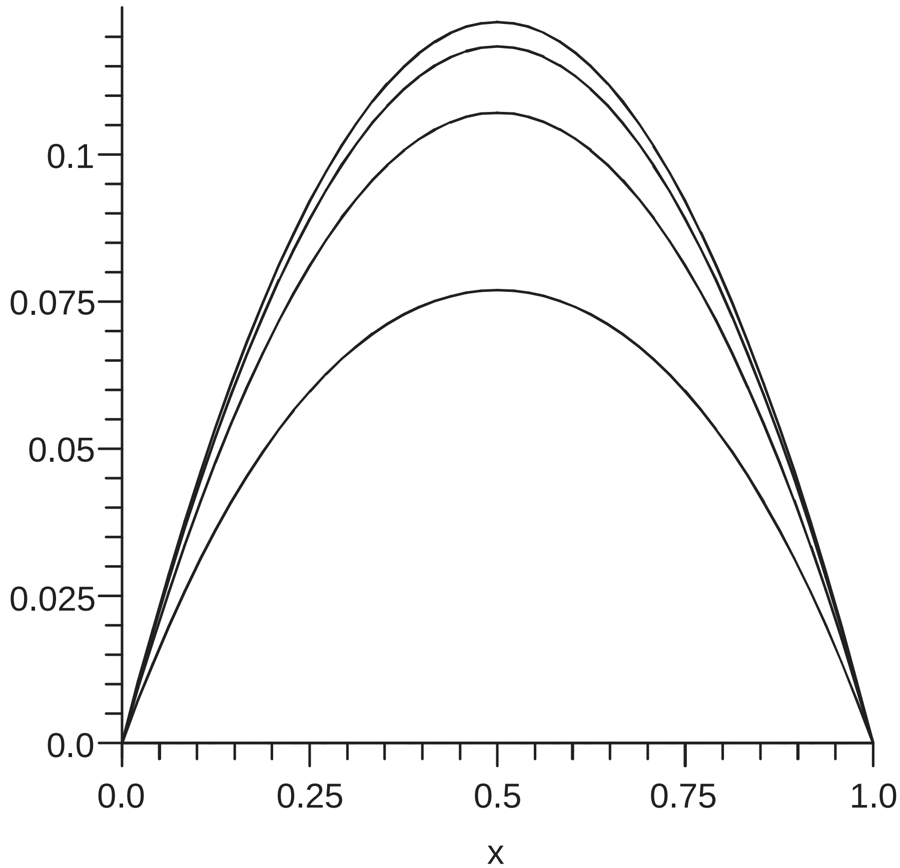
>

Here are pictures

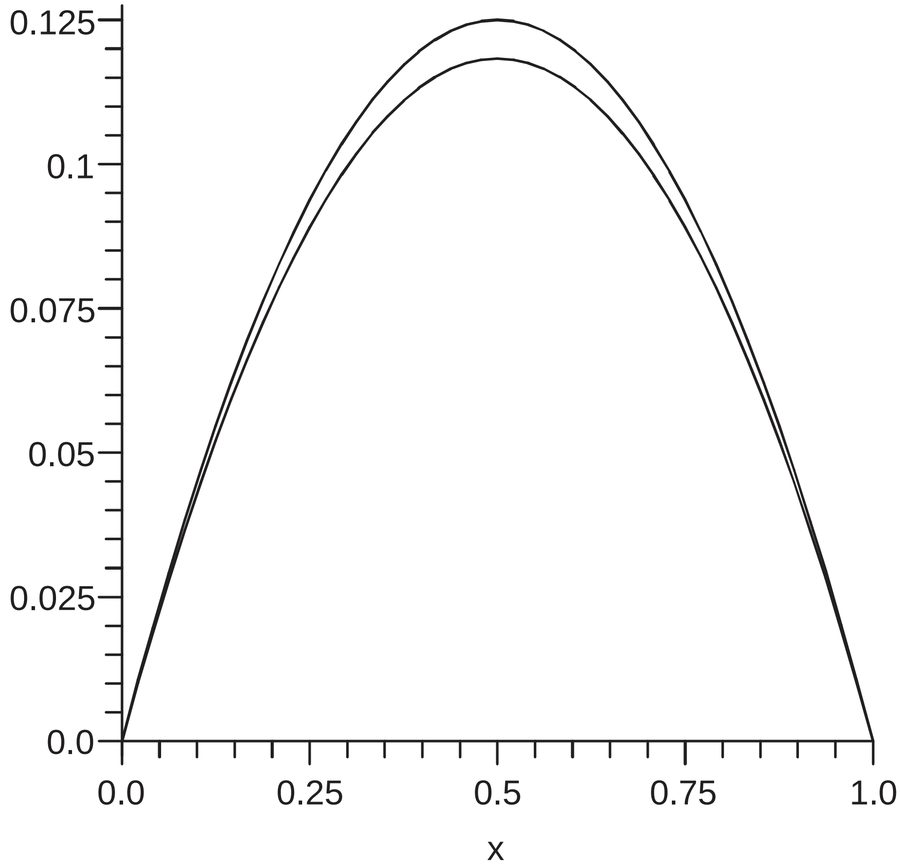
> $plot([seq(u1N(x, k\cdot.01, 25), k = 0..4)], x = 0..1, color = black);$



> $plot([seq(u1N(x, k\cdot.1, 25), k = 0..4)], x = 0..1, color = black);$



> *plot*([*v*(*x*), *u1N*(*x*,.3, 25)], *x* = 0 ..1, *color* = *black*);



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Example 2 - HE with convective BC

uxx = ut   0 < x < 1, 0 < t
u(0,t) = 50
-ux(1,t) = (u(1,t) - 100)
u(x,0) = 0

The steady state solution is

```
> v2 := x → 25 · x + 50;
```

$$v2 := x \rightarrow 25\,x + 50 \tag{2.1}$$

>

Subtracting the steady-state solution, we obtain the problem with homogeneous boundary values

wxx = wt
w(0,t) = 0

w(1,t) + wx(1,t) = 0
w(x,0) = g(x)

with

> $g2 := x \rightarrow -\ v2(x);$

$$g2 := x \rightarrow -v2(x) \quad (2.2)$$

The eigenvalues are - lambdan^2 and eigenfunctions are as follows. The first few Lamnbas are found in our text, the rest are from Abramowitz and Stegun.

> $X := (x, n) \rightarrow sin(lambda(n) \cdot x);$

$$X := (x, n) \rightarrow \sin(\lambda(n)\, x) \quad (2.3)$$

>

> $lambda(1) := 2.0288;\ lambda(2) := 4.9132;$
> $lambda(3) := 7.9787;\ lambda(4) := 11.0855;$
> $lambda(5) := 14.2074;\ lambda(6) := 17.3364;$
> $lambda(7) := 20.4692;\ lambda(8) := 23.6043;$
> $lambda(9) := 26.7409;$

$$\lambda(1) := 2.0288$$
$$\lambda(2) := 4.9132$$
$$\lambda(3) := 7.9787$$
$$\lambda(4) := 11.0855$$
$$\lambda(5) := 14.2074$$
$$\lambda(6) := 17.3364$$
$$\lambda(7) := 20.4692$$
$$\lambda(8) := 23.6043$$
$$\lambda(9) := 26.7409 \quad (2.4)$$

The coefficients of the series solution for w are

> $b2 := n \rightarrow \frac{(evalf(Int(g2(x) \cdot sin(lambda(n) \cdot x), x = 0..1)))}{int\left(sin((lambda(n) \cdot x))^2, x = 0..1\right)};$

$$b2 := n \rightarrow \frac{evalf\left(\int_0^1 g2(x)\sin(\lambda(n)\, x)\, dx\right)}{\int_0^1 \sin(\lambda(n)\, x)^2\, dx} \quad (2.5)$$

>

and the 9th partial sum is

> $w2 := (x, t) \rightarrow add\left(b2(n) \cdot X(x, n) \cdot exp\left(-lambda(n)^2 \cdot t\right), n = 1..9\right);$

$$w2 := (x, t) \rightarrow add\left(b2(n)\, X(x, n)\, e^{\left(-\lambda(n)^2 t\right)}, n = 1..9\right) \quad (2.6)$$

>

The 9th partial sum for u is

```
> u2 := (x, t) → v2(x) + w2(x, t);
```

$$u2 := (x, t) \rightarrow v2(x) + w2(x, t) \tag{2.7}$$

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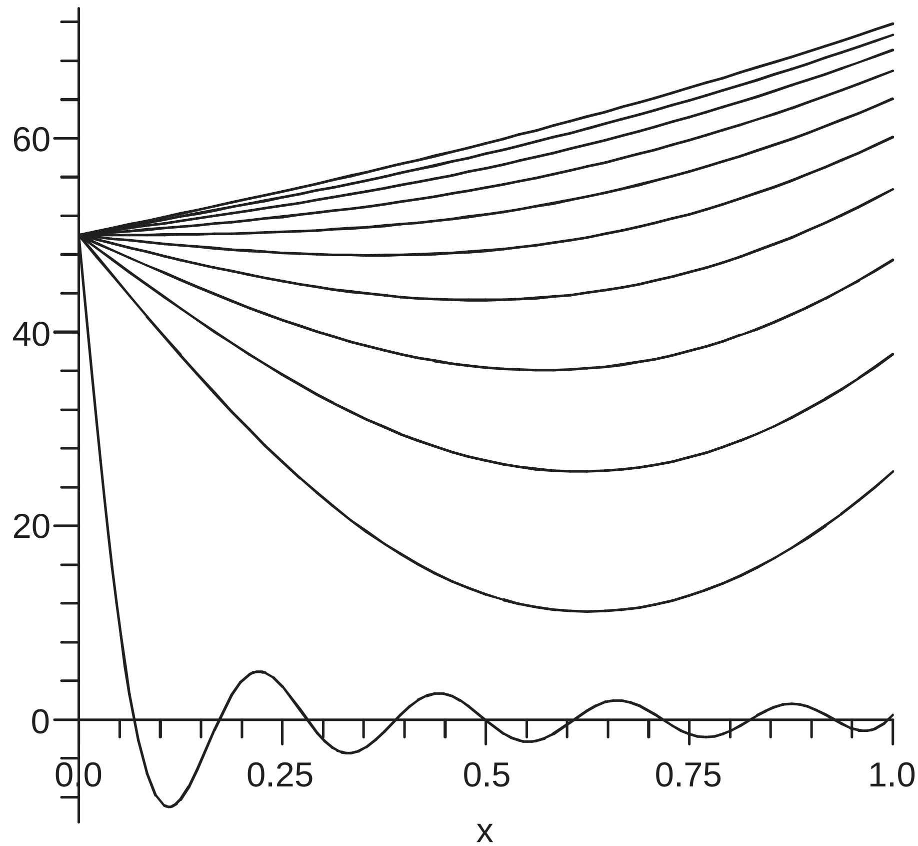
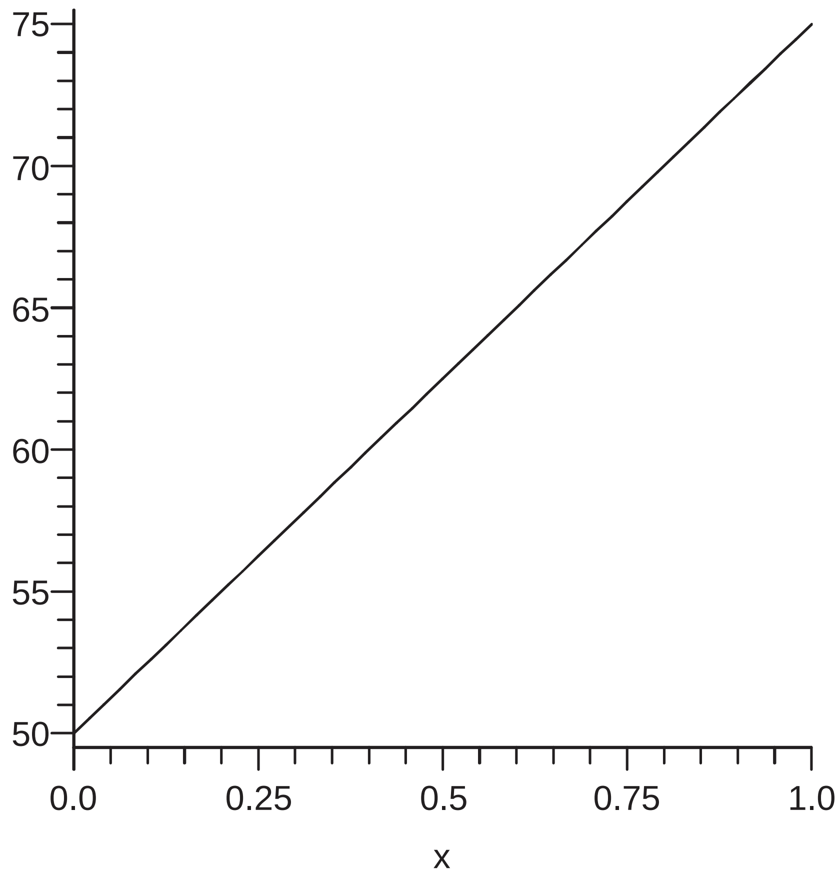
>
```

As usual, here are pictures showing the solution for various small values of t, followed by the steady state solution.

```
> plot( [seq(u2(x, k·.075), k=0..10)], x=0..1,
      color = black);
```




```
>
>
> plot(u2(x, 15), x=0..1, color=black);
```


75
70
65
60
55
50
0.0
0.25
0.5
0.75
1.0
x

>