The Wave Equation

in a rectangle

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Math 4581

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Normal Modes

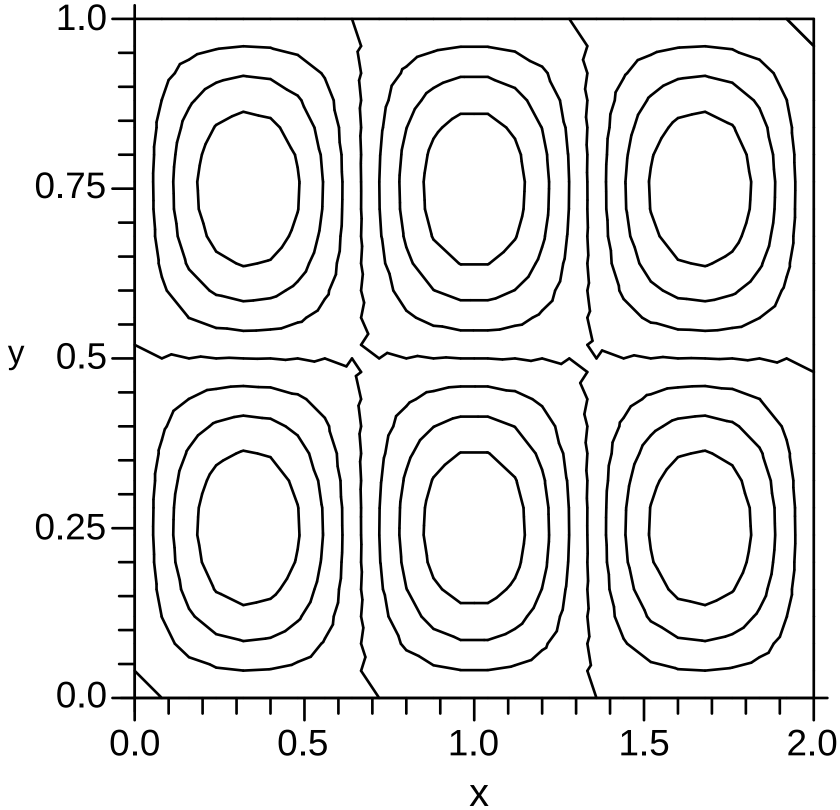
Here we will illustrate one of the normal modes on the rectangle [0,2] X [0,1].

> $with(plots):$

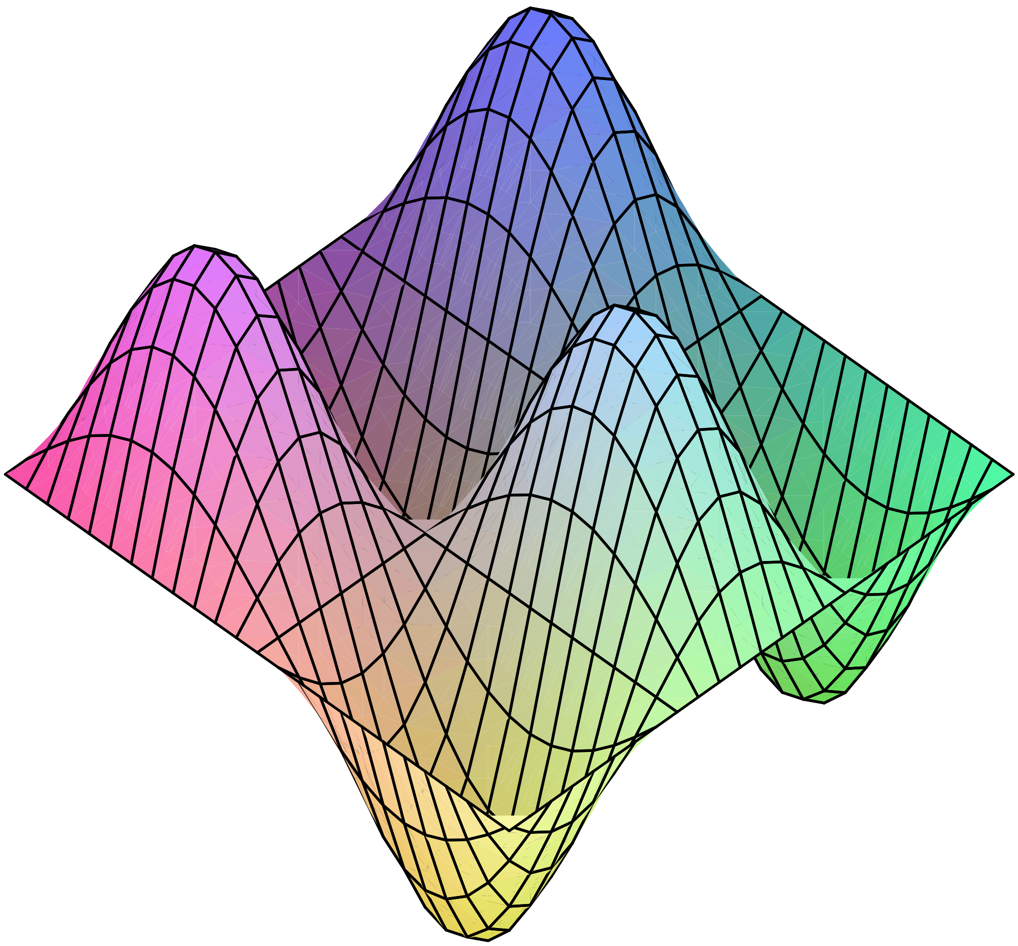
> $u32 := (x, y, t) \to \sin\left(\frac{3\cdot\pi}{2}\cdot x\right)\cdot\sin(2\cdot\pi\cdot y)\cdot\cos\left(\frac{5\cdot\pi}{2}\cdot t\right);$

$$u32 := (x, y, t) \to \sin\left(\frac{3}{2}\,\pi\,x\right)\,\sin(2\,\pi\,y)\,\cos\left(\frac{5}{2}\,\pi\,t\right) \qquad (1.1)$$

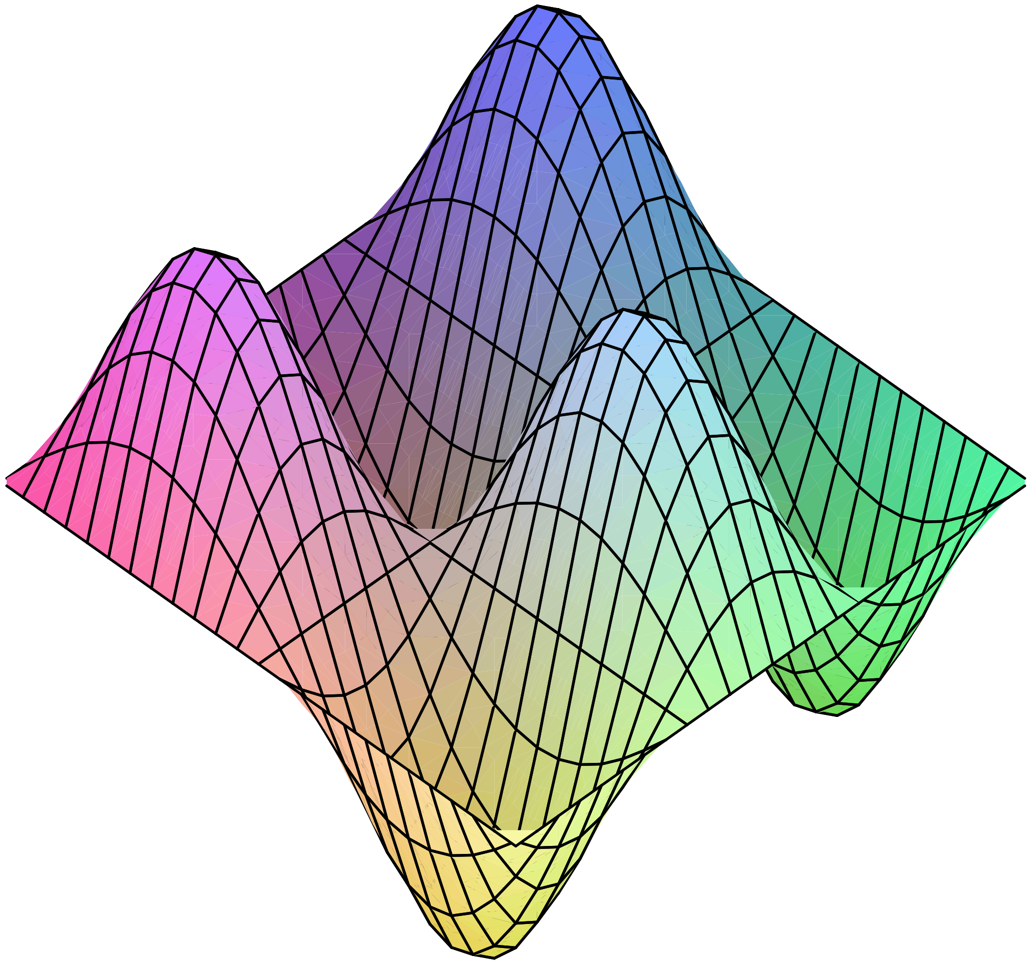
> $contourplot(u32(x, y, 0), x = 0..2, y = 0..1, color = black,$
> $contours = [-.75, -.5, -.25, 0, .25, .5, .75]);$



> $plot3d(u32(x, y, 0), x = 0..2, y = 0..1);$



> $animate\left(plot3d, [u32(x, y, t), x = 0..2, y = 0..1], t = 0..\frac{4}{5}\right);$



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Superposition of Two Normal Modes

Here we look at the superposition of two normal modes on the unit square

> *with*(*plots*) :

> $u34 := (a, b, x, y, t) \rightarrow \sin\left(3\cdot\pi\cdot\frac{x}{a}\right)\cdot\sin\left(\frac{4\cdot\pi\cdot y}{b}\right)\cdot\cos\left(\pi\cdot\mathrm{sqrt}\left(\left(\frac{3}{a}\right)^2+\left(\frac{4}{b}\right)^2\right)\cdot t\right);$

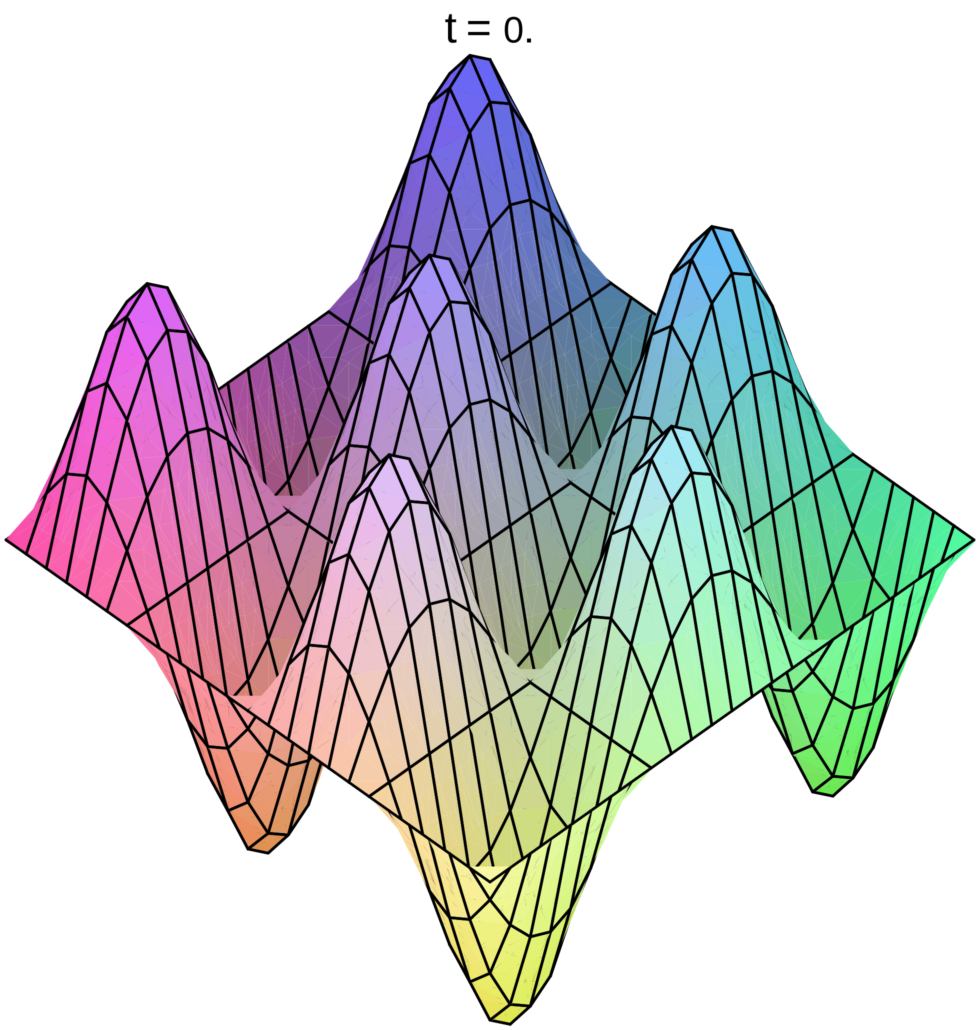
$$u34 := (a, b, x, y, t) \rightarrow \sin\left(\frac{3\,\pi\,x}{a}\right)\sin\left(\frac{4\,\pi\,y}{b}\right)\cos\left(\pi\sqrt{\frac{9}{a^2}+\frac{16}{b^2}}\;t\right) \tag{2.1}$$

> $u512 := (a, b, x, y, t) \rightarrow \sin\left(\frac{5\cdot\pi\cdot x}{a}\right)\sin\left(\frac{12\cdot\pi\cdot y}{b}\right)\cdot\cos\left(\pi\cdot\mathrm{sqrt}\left(\left(\frac{5}{a}\right)^2+\left(\frac{12}{b}\right)^2\right)\cdot t\right);$

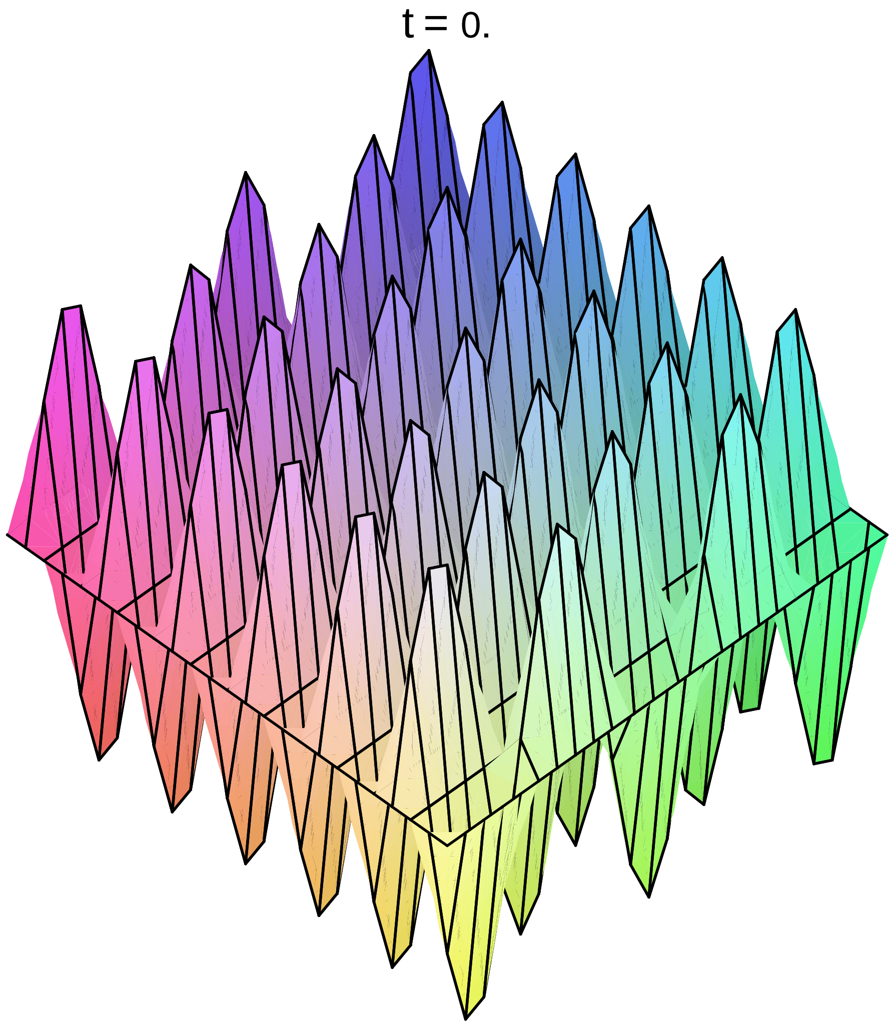
(2.2)

$$u512 := (a, b, x, y, t) \rightarrow \sin\left(\frac{5\pi x}{a}\right)\sin\left(\frac{12\pi y}{b}\right)\cos\left(\pi\sqrt{\frac{25}{a^2}+\frac{144}{b^2}}\,t\right) \tag{2.2}$$

> $animate\left(plot3d, [u34(1, 1, x, y, t), x = 0..1, y = 0..1], t = 0..\frac{2}{5}\right);$



> $animate\left(plot3d, [u512(1, 1, x, y, t), x = 0..1, y = 0..1], t = 0..\frac{2}{13}\right);$



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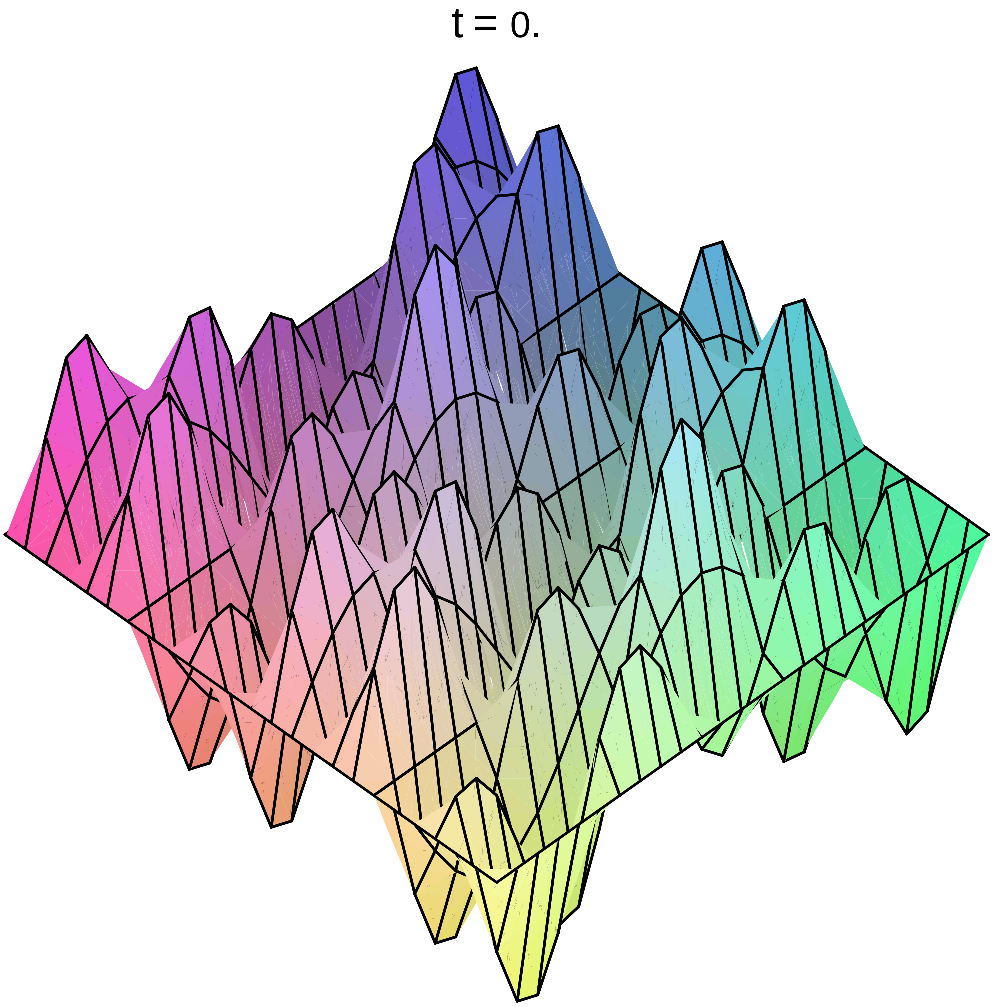
We now animate the sum from t = 0 to 2. Thus we see 5 periods of one and 13 periods of the other.

> $animate\left(plot3d, \left[u34(1, 1, x, y, t) + u512(1, 1, x, y, t), x = 0..1, y = 0..1\right], t = 0..2, frames = 100\right);$



Now we'll change the shape of the rectangle so that the motion is not periodic.
We'll continue to let b = 1, and choose a = 2

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> animate(plot3d, [u30(sqrt(3), 1, x, y, t) + u02(sqrt(3), 1, x, y, t), x = 0..2,
      y = 0..1], t = 0..20, frames = 100);
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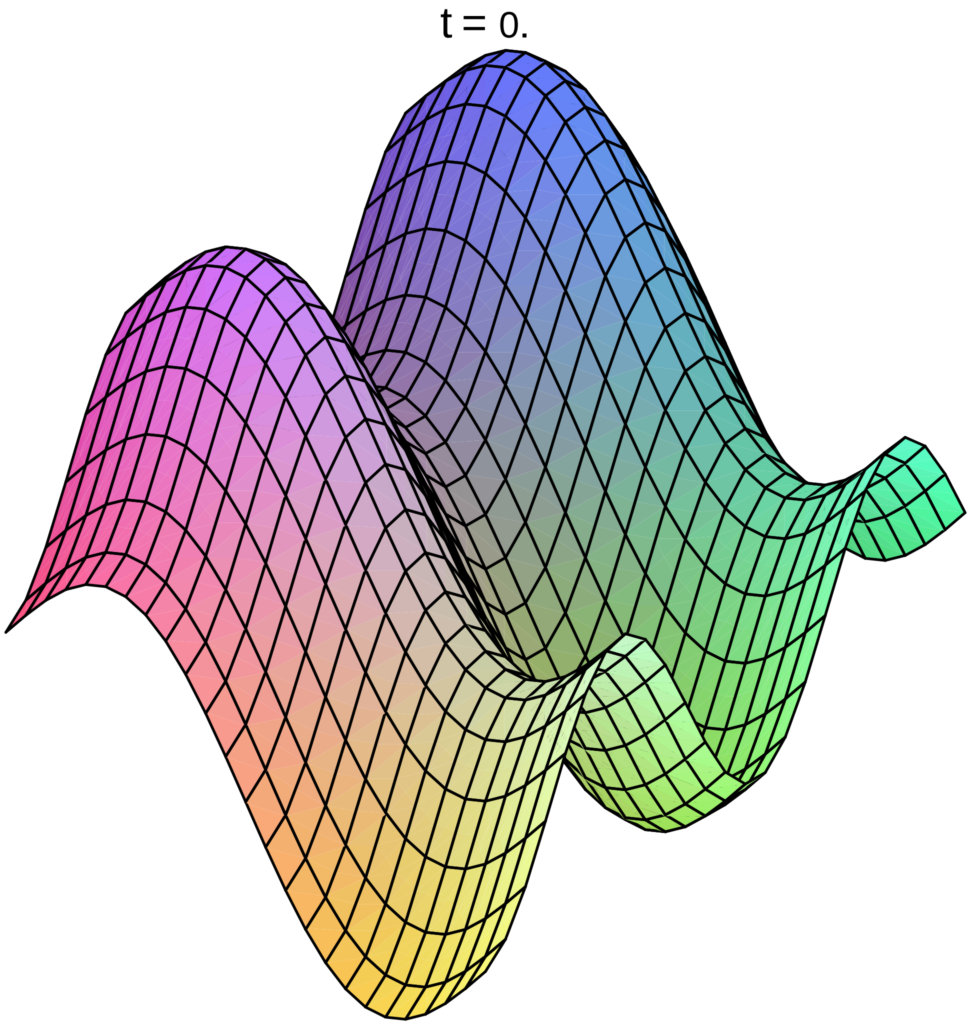

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A more general situation

Here we will continue to use the rectangle [0,2]x[0,1] with u = 0 on the boundaries. The initial velocity will be 0, and the initial position will be a small bump in the square [1/4,3/4] X [1/4,3/4]. Here is the bump.

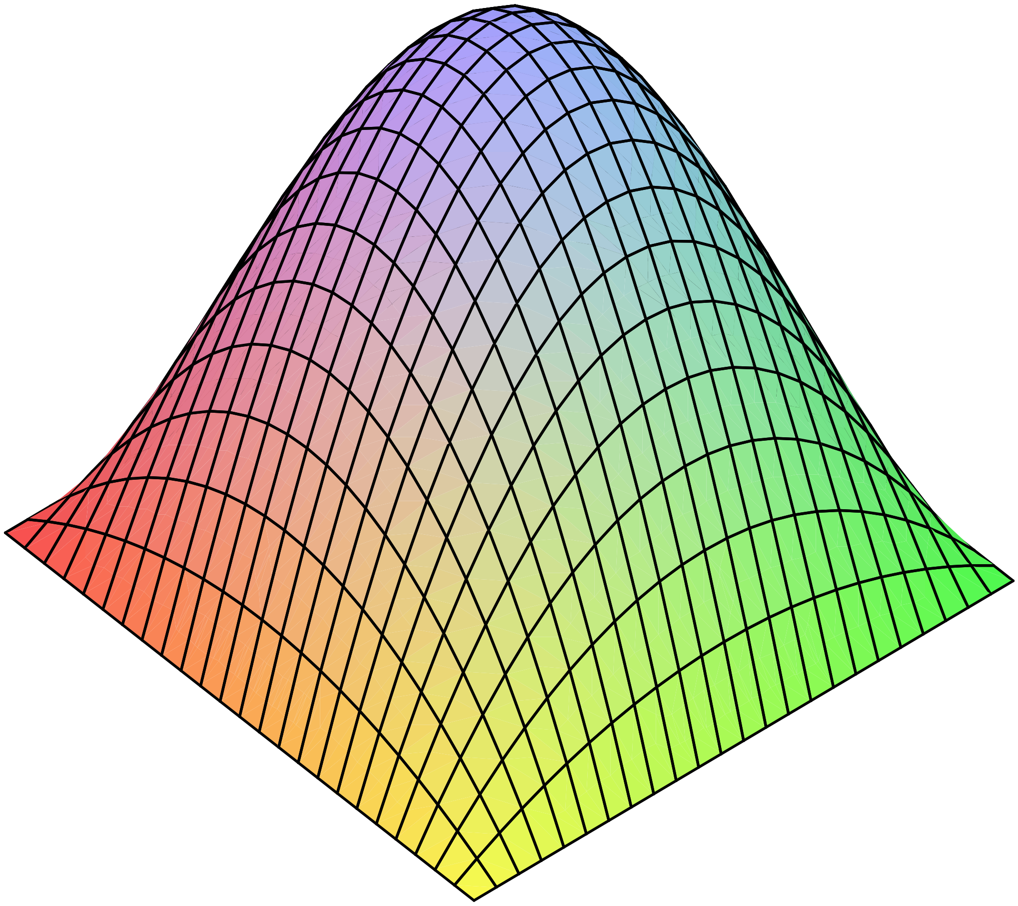
> $f1 := x \rightarrow \left(x - \frac{1}{4}\right) \cdot \left(\frac{3}{4} - x\right);$

$$f1 := x \rightarrow \left(x - \frac{1}{4}\right) \left(\frac{3}{4} - x\right) \quad (3.1)$$

> $f := (x, y) \rightarrow f1(x) \cdot f1(y);$

$$f := (x, y) \rightarrow f1(x)\, f1(y) \quad (3.2)$$

> $plot3d(f(x, y), x = .25..0.75, y = .25..0.75);$



The functions X, and Y that result from separation of variables are

> $X := (x, m) \to \sin\left(\frac{m \cdot \pi \cdot x}{2}\right);$

$$X := (x, m) \to \sin\left(\frac{1}{2}\,\pi\,m\,x\right) \tag{3.3}$$

> $Y := (y, n) \to \sin(n \cdot \pi \cdot y);$

$$Y := (y, n) \to \sin(\pi\,n\,y) \tag{3.4}$$

The Fourier Coefficients of the initial position are then

> $a := (m, n) \to 2 \cdot evalf\left(Int\left(evalf(Int(f(x, y) \cdot X(m, x) \cdot Y(n, y),\ x = .25..0.75)),\ y = .25..0.75\right)\right);$

$$a := (m, n) \to 2\ evalf\left(\int_{0.25}^{0.75} evalf\left(\int_{0.25}^{0.75} f(x, y)\ X(m, x)\ Y(n, y)\,\mathrm{d}x\right)\mathrm{d}y\right) \tag{3.5}$$

> $a(5, 6);$

$$9.045279344\ 10^{-15} \tag{3.6}$$

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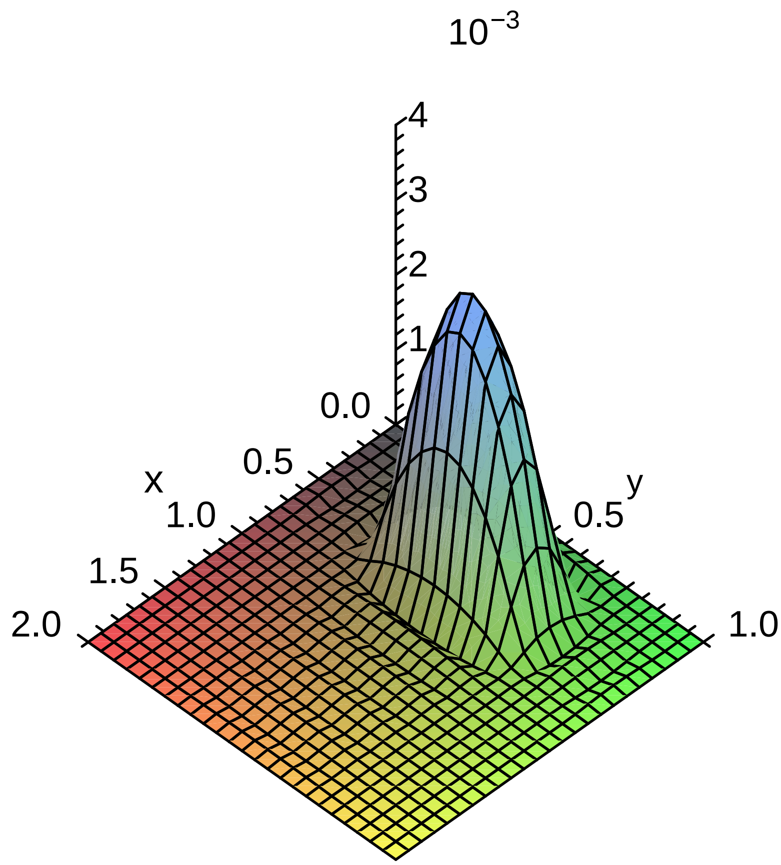
An approximate solution to our boundary value problem is then

> $u3 := (x, y, t) \rightarrow add\left(add\left(a(m, n)\cdot X(x, m)\cdot Y(y, n)\cdot \cos\left(\pi\cdot \mathrm{sqrt}\left(\frac{m^2}{4} + n^2\right)\cdot t\right), m = 1..15\right), n = 1..15\right):$

> $plot3d(u3(x, y, 0), x = 0..2, y = 0..1, axes = normal);$



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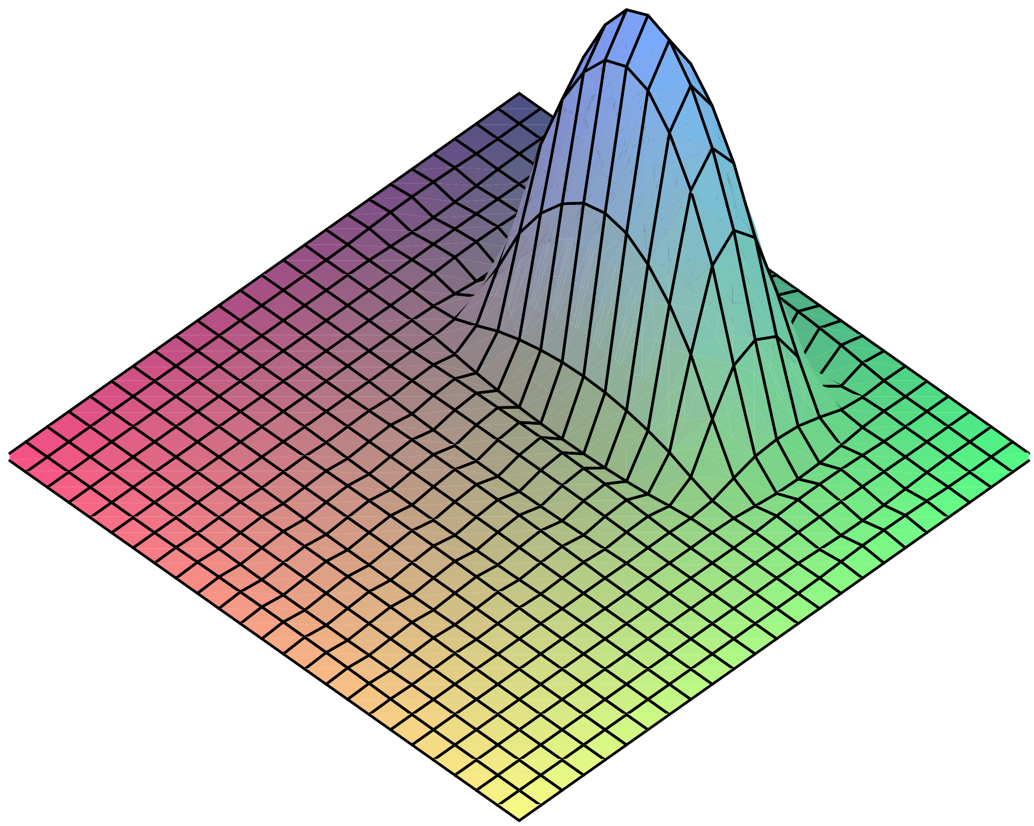
> $animate(plot3d, [u3(x, y, t), x = 0..2, y = 0..1], t = 0..4);$

t = 0.



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