Homework assignments for Math 4221 Spring 2008

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1 Due by February 6 2008

1. Consider the Bernoulli scheme for some $n \in \mathbb{N}$, $p \in (0,1)$ and $q = 1-p$:

$$\Omega = \{(\omega_1, \dots, \omega_n) : \quad \omega_k = 0 \text{ or } 1 \text{ for all } k\},$$
$$p(\omega) = (p^{\omega_1} q^{1-\omega_1}) \dots (p^{\omega_n} q^{1-\omega_n}),$$
$$X_k(\omega) = \omega_k,$$

   Prove that $\mathsf{P}\{X_2 = 1\} = p$.

2. Let $X_1, X_2, \dots$ be Bernoulli random variables with parameter $p \in (0,1)$. Let $S_n = X_1 + \dots + X_n$. Find $\mathsf{P}\{S_3 = 2\}$.

3. Let us consider a tetrahedral die. Three of its faces are painted Red, Green, and Blue, respectively, and the fourth is painted with all three colors. We roll the die once and look at the bottom face. We define

$$X_R = \begin{cases} 1, & \text{there is Red on the bottom face} \\ 0, & \text{there is NO Red on the bottom face} \end{cases},$$

   and random variables X_G, X_B are defined similarly. We assume that each of the four faces has probability $1/4$. Prove that two random variables in each pair (X_R, X_G), (X_R, X_B), and (X_G, X_B) are independent of each other. Prove that X_R, X_G, X_B are not jointly independent.

4. Let $X_1, X_2, \dots$ be Bernoulli random variables with parameter $p \in (0,1)$. Let $S_n = X_1 + \dots + X_n$. Prove that S_3 is not independent of X_1.

5. Let $X_1, X_2, \dots$ be independent identically distributed (i.i.d.) random variables (r.v.'s) with

$$\mathsf{P}\{X_1 = 1\} = p, \quad \mathsf{P}\{X_1 = -1\} = q.$$

Let S_n be the random walk generated by X:

$$S_n = X_1 + \ldots + X_n.$$

For any n, find a formula for $\mathsf{P}\{S_n = 0\}$.

6. In the setting of the previous problem, let us take any whole numbers $A < B$ and for each $x \in \{A, A+1, \ldots, B\}$ define $S_n^x = x + S_n$. Let

$$\beta(x) = \mathsf{P}\{\text{random walk } S^x \text{ reaches level } B \text{ before reaching level } A\}.$$

Draw a sketch of the graph of $\beta(x)$ for three cases (a)$p > q$, (b)$p = q = 1/2$, (c)$p < q$.

7. In the setting the previous problem, fix B and study the limiting behaviour of $\beta(x)$ as $A \to -\infty$.

2 Due by February 20 2008

In this set of problems we consider a symmetric random walk: $(X_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. r.v.'s with $\mathsf{P}\{X_1 = 1\} = \mathsf{P}\{X_1 = -1\} = 1/2$, and $S_n = X_1 + \ldots + X_n$ for all n. We also set $S_0 = 0$.

1. Use Local Central Limit Theorem to derive

$$\mathsf{P}\{S_{2k} = 0\} \sim \frac{1}{\sqrt{\pi k}}, \quad k \to \infty.$$

2. Use the reflection principle to show that for any $N > 0$

$$\mathsf{P}\left\{\max_{1\le k\le n} S_k \ge N, \quad S_n < N\right\} = \mathsf{P}\{S_n > N\}.$$

3. Prove that for any $N > 0$,

$$\mathsf{P}\left\{\max_{1\le k\le n} S_k \ge N\right\} = 2\mathsf{P}\{S_n \ge N\} - \mathsf{P}\{S_n = N\}.$$

4. Prove that for any $N > 0$,

$$\mathsf{P}\left\{\max_{1\le k\le n} S_k = N\right\} = \mathsf{P}\{S_n = N\} + \mathsf{P}\{S_n = N+1\}.$$

5. Fix any $N > 0$. Find

$$\mathsf{P}\left\{\max_{1\le k\le n} S_k \ge N, \quad S_n = 0\right\}$$

6. Let

$$g_{2n} = \max\{0 \le 2k \le 2n : \ S_{2k} = 0\},$$

(i.e. g_{2n} is the number of the last zero in the sequence $(S_0, S_2, S_4, \ldots, S_{2n})$)

Prove that

$$\mathsf{P}\{g_{2n} = 2k\} = u_{2k}u_{2(n-k)}, \tag{1}$$

where $u_{2k} = \mathsf{P}\{S_{2k} = 0\}$.

7. In the setting of the previous problem use relation (1) to formulate and prove an arcsine law for g_{2n}.

8. Let

$$\tau_1 = \min\{n > 0 : \ S_n = 0\}.$$

This is the time of first return to 0. Define also

$$\tau_2 = \min\{n > \tau_1 : \ S_n = 0\}$$

and

$$\tau_3 = \min\{n > \tau_2 : \ S_n = 0\},$$

the times of second and third return.

(a) Prove that τ_1 and $\tau_2 - \tau_1$ are independent random variables.

(b) Prove that $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2$ are jointly independent.

3 Due by Wednesday April 2 2008

1. Let $\mathsf{P}\{X = k\} = (1 - p)p^k$, for some $p > 0$ and all $k = 0, 1, \ldots$. Find the generating function of X and use it to find $\mathsf{E}X$ and $\mathsf{Var}X$.

2. Let X_1 and X_2 be two independent Poisson r.v.'s with parameters λ_1 and λ_2, respectively. Use generating functions to find the distribution of $X_1 + X_2$.

3. Consider an asymmetric random walk $S_n = X_1 + \ldots + X_n$, where (X_k) is a sequence of i.i.d. r.v.'s with $\mathsf{P}\{X_k = 1\} = p \neq \mathsf{P}\{X_k = -1\} = 1 - p$. Let $\tau = \min\{n \geq 0 : \ S_n = 1\}$. Find the generating function of τ. (Use the same logic that we used in class when considering the symmetric case).

4. In the setting of the previous problem, let $\tilde{\tau} = \min\{n > 0 : \ S_n = 0\}$. Use the result of the previous problem to find the generating function of $\tilde{\tau}$. Use it (i) to find $\mathsf{P}\{\tilde{\tau} = k\}$ for all k; (ii) to prove that $\sum_{k>0} \mathsf{P}\{\tilde{\tau} = k\} < 1$.

5. Consider a symmetric "lazy" random walk: $S_n = S_n = X_1 + \ldots + X_n$, where (X_k) is a sequence of i.i.d. r.v.'s with $\mathsf{P}\{X_k = 1\} = \mathsf{P}\{X_k = -1\} = \mathsf{P}\{X_k = 0\} = 1/3$. Let $\tau = \min\{n \geq 0 : \ S_n = 1\}$. Find the generating function of τ.

6. In the setting of the previous problem, let $\tilde{\tau} = \min\{n > 0 : \ S_n = 0\}$. Use the result of the previous problem to find the generating function of $\tilde{\tau}$. Use it to prove that $\sum_{k>0} \mathsf{P}\{\tilde{\tau} = k\} = 1$.

7. Prove that for a Markov chain with finitely many states there is at least one state that is not inessential. Is this true for Markov chains with infinitely many states?

8. Prove that for asymmetric random walk, the origin is transient.

9. Prove that any two communicating states of a Markov chain are either both transient, or both recurrent.

10. For each of the following Markov transition matrices, (i) draw the associated transition graph; (ii) find classes of communicating states and inessential states; (iii) for each class find its period; (iv) find all stationary distributions; (v) for each initial state consider the Markov chain with given transition probabilities and originatig at that state;

does the distribution of this Markov chain at time n have a limit as $n \to \infty$? do the limits coincide for different initial states? If your answer to at least one of these questions is "no", explain why this does not contradict the Perron–Frobenius theorem.

(a) "Symmetric random walk with absorption"

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) "Symmetric random walk with reflection"

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

4 Due by April 29

In first 4 problems we consider a branching process (X_n) with $X_0 = 1$ and branching distribution $(P_j)_{j=0}^{\infty}$.

1. Let $P_0 = P_2 = 1/2$. Find the extinction probability Q.
2. Let $P_0 = 1/3$, $P_2 = 2/3$. Find the extinction probability Q.
3. Let $P_0 = P_2 = 1/2$. Find $P_{ij} = \mathsf{P}\{X_{n+1} = j | \ X_n = i\}$ for all i, j.
4. Let $G_n(s) = \mathsf{E}s^{X_n}$ and $G(s) = G_1(s) = \mathsf{E}s^{X_1}$. Prove that

$$G_n(s) = \underbrace{G(G(\dots G}_{n}(s)\dots))$$

5. Any 1 problem out of 4.34, 4.55(a) from the book
6. Any 1 problem out of 4.39, 4.41, 4.44, 4.47(with $m = 2$).
7. Any 4 problems out of 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.10, 4.13, 4.31