Homework assignments for Math 6242 Fall 2006

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1 Due on September 11 2006

1. Prove that if X and Y are independent r.v.'s then for any bounded complex-valued functions f and g
$$\mathbb{E}f(X)g(Y) = \mathbb{E}f(X)\mathbb{E}g(Y).$$

2. Compute characteristic functions for examples (1–8) and (11),(12) on p.155–156 of the textbook. Hint: For (12) use residues.

3. Let $x, h \in \mathbb{R}$. Prove that
$$\left|\frac{e^{ihx} - 1}{h}\right| \le |x|.$$

4. Let X_λ be a λ-Poisson r.v. for every positive λ. Find the limits:
$$\lim_{\lambda\to\infty} \phi_{\frac{X_\lambda}{\lambda}}(t),$$
$$\lim_{\lambda\to\infty} \phi_{\frac{X_\lambda-\lambda}{\sqrt{\lambda}}}(t),$$

5. Use characteristic functions to compute $\mathbb{E}X^k, k = 1, 2, 3, 4$ if X is
   (a) an $N(0, \sigma^2)$ - r.v.
   (b) a λ-Poisson r.v., $\lambda > 0$
   (c) an $\exp(\lambda)$ -r.v., $\lambda > 0$

6. Prove the weak Law of Large Numbers: Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with $\mathbb{E}|X_1| < \infty$ and show that for all $t \in \mathbb{R}$
$$\lim_{n\to\infty} \phi_{\frac{X_1+\ldots+X_n}{n}}(t) = e^{imt},$$
where $m = \mathbb{E}X_1$.

7. Problem 1 from Section 6.4 of the book.

2 Due on September 22 2006

1. Compute characteristic functions for examples (9),(10) on p.156 of the textbook.

2. Suppose that there is a number a such that a r.v. X takes values only of the form $ka, k \in \mathbb{Z}$. Prove that the charactereistic function ϕ_X is periodic and find the smallest period.

3. Let ϕ be the ch.f. of a distribution $P(dx)$. Prove that for each x_0 we have
$$\lim_{c\to\infty} \frac{1}{2c} \int_{-c}^{c} e^{-itx_0}\phi(t)dt = P(\{x_0\}).$$

4. Suppose a r.v. X takes values in $\mathbb{Z}$. Prove
$$\mathsf{P}\{X = k\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt}\phi_X(t)dt.$$

5. Solve Problem 7 from Section 6.2 of the book, p.167.

6. Suppose
$$\ln \phi_X(t) = \sum_{k=1}^{m} \frac{s_X(k)}{k!}(it)^k + o(|t|^m)$$
Prove that if X and Y are independent r.v.'s, then
$$s_{X+Y}(k) = s_X(k) + s_Y(k).$$

7. Give an example of a sequence of ch.f.'s ϕ_n such that $\phi_n(t) \to \phi_\infty(t)$ for all t, but ϕ_∞ is not continuous at 0.

8. Use characteristic functions to prove Poisson's theorem: Suppose for any $n \geq 1$ we have n independent r.v.'s $\xi_{n1}, \xi_{n2}, \ldots, \xi_{nn}$. Each of these r.v.'s takes two values:
$$\mathsf{P}\{X_{nk} = 1\} = p_{nk}, \quad \mathsf{P}\{X_{nk} = 0\} = q_{nk}, \quad p_{nk} + q_{nk} = 1.$$
Assume that $\max_{1\leq k\leq n} p_{nk} \to 0, \quad p_{n1} + p_{n2} + \ldots + p_{nn} \to \lambda > 0$. If
$$S_n = \xi_{n1} + \xi_{n2} + \ldots + \xi_{nn},$$
then
$$\lim_{n\to\infty} \mathsf{P}\{S_n = m\} = \frac{e^{-\lambda}\lambda^m}{m!}.$$

3 Due on October 18 2006

1. Suppose that $(Z_n)_{n\in\mathbb{Z}}$ is a stationary process, and $f:\mathbb{R}^d \to \mathbb{R}$ is a Borel-measurable function. Prove that $(X_n)_{n\in\mathbb{Z}}$ is also a stationary process if $X_n = f(Z_n,\dots,Z_{n+d-1})$.

2. Give an example of a stationary m-dependent process X with the following properties: (i)$\mathsf{P}\{X_0=0\}<1$, (ii)$\mathbb{E}X_0=0$ (iii) $\mathbb{E}(X_1+\ldots+X_n)^2=o(n)$, (iv) CLT does not hold for $X_1+\ldots+X_n$.

3. Prove Slutskii's lemma: Suppose $X_n \stackrel{dist}{\to} X_\infty$, and $Y_n \stackrel{\mathsf{P}}{\to} 0$, $Z_n \stackrel{\mathsf{P}}{\to} 1$. Then $X_n+Y_n \stackrel{dist}{\to} X_\infty$, $X_nZ_n \stackrel{dist}{\to} X_\infty$.

4. Suppose $(X_j)_{j\in\mathbb{Z}^2}$ is a family of random variables indexed by the lattice $\mathbb{Z}^2$. Suppose $\mathbb{E}X_j=0$ for all $j\in Z^2$, and $\mathbb{E}X_kX_j=r(k-j)$ for all $j,k\in\mathbb{Z}^2$. Suppose further that there is a number m such that $r(j)=0$ if $|j|>m$. Here $|j|=\max\{|j_1|,|j_2|\}$.

   Let
   $$S_n=\sum_{0<j_1,j_2\le n} X_j.$$
   Prove that there is a constant $\Gamma\ge 0$ such that
   $$ES_n^2=\Gamma n^2+o(n^2).$$

5. Suppose $(X_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. r.v.'s with $\mathbb{E}X_1=0$, $\mathbb{E}X_1^2=1$. Find the limit distribution of
   $$\frac{X_1+\ldots+X_n}{\sqrt{X_1^2+\ldots+X_n^2}}.$$

6. Show that if ϕ_1 and ϕ_2 are infinitely divisible characteristic functions, then $\phi_1\phi_2$ is also infinitely divisible.

7. Let ϕ_n be an infinitely divisible characteristic function for each $n\in\mathbb{N}$, and let $\phi_n(t)\to\phi_\infty(t)$ as $n\to\infty$ for all t. Show that ϕ_∞ is infinitely divisible.

8. Suppose $\mu(\cdot)$ is a finite measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. Show that
   $$f(t)=\exp\left\{\int_{\mathbb{R}}(e^{itu}-1)\mu(du)\right\}$$
   is an infinitely divisible characteristic function.

9. Show that the uniform distribution is not infinitely divisible.

4 Due on November 3 2006

1. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. standard Gaussian r.v.'s. Show that
$$\mathsf{P}\left\{\limsup_{n\to\infty}\frac{X_n}{\sqrt{2\ln n}}=1\right\}=1.$$

2. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. λ-Poisson r.v.'s. Show that
$$\mathsf{P}\left\{\limsup_{n\to\infty}\frac{X_n\ln\ln n}{\ln n}=1\right\}=1.$$

3. Let Y be a r.v. measurable w.r.t. the tail σ-algebra for independent r.v.'s $X_1, X_2, \ldots$. Prove that Y is degenerate, i.e. there is a constant c such that $\mathsf{P}\{Y=c\}=1$.

4. Suppose $X_1, X_2, \ldots$ is a sequence of independent r.v.'s. Prove that $\limsup_{n\to\infty} X_n$ is degenerate.

5. Let $S_n = X_1 + \ldots + X_n$ and $\mathcal{G} = \bigcap_n \mathcal{G}_{[n,\infty)}$ where $\mathcal{G}_{[n,\infty)} = \sigma\{S_n, S_{n+1}, \ldots\}$. Show that each event in $\mathcal{G}$ is permutable.

6. Give an example of a sequence of r.v.'s and a tail event that has probability strictly between 0 and 1. To make it interesting, insist that the r.v.'s are not identical.

7. Suppose $(X_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence in $\mathbb{R}^1$ with X_1 taking values ± 1 with probability $1/2$. Is 0 a point of recurrence?

8. Suppose $(X_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence in $\mathbb{R}^2$ with X_1 taking values $(\pm 1, \pm 1)$ with probability $1/4$. Is $(0,0)$ a point of recurrence?

9. Suppose $(X_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence in $\mathbb{R}^3$ with X_1 taking values $(\pm 1, \pm 1, \pm 1)$ with probability $1/8$. Is $(0,0,0)$ a point of recurrence?

5 Due on December 6 2006

1. Suppose that $\mathcal{D} = \{D_1, \ldots, D_n\}$ is a partition of a probability space $\{\Omega, \mathcal{F}, \mathsf{P}\}$. Let X be a square integrable r.v. Find the vector $(a_1, \ldots, a_n)$ that solves

$$\mathbb{E}\left[\sum_{i=1}^{k}(a_i - X)^2 \mathbf{1}_{D_i}\right] \to \min$$

and express it in terms of $\mathbb{E}(X|\mathcal{D})$.

2. Let X and Y be i.i.d. with $\mathbb{E}|X| < \infty$. Prove that

$$\mathbb{E}(X|X+Y) = \frac{X+Y}{2} \quad \text{a.s.}$$

3. Let $(X_k)_{k\in\mathbb{N}}$ be an i.i.d. sequence. Show that

$$\mathbb{E}(X_1|S_n, S_{n+1}, \ldots) = \frac{S_n}{n} \quad \text{a.s.,}$$

where $S_n = X_1 + \ldots + X_n$.

4. Prove Jensen's inequality for conditional expectations: Suppose $\mathbb{E}|X| < \infty$ and $\mathbb{E}|g(X)| < \infty$ for a convex function g (the graph of g is U-shaped). Then

$$g(\mathbb{E}[X|\mathcal{G}]) \le \mathbb{E}[g(X)|\mathcal{G}].$$

5. Suppose a 2-component random vector (X, Y) has a density $p_{X,Y}(x, y)$ w.r.t. Lebesgue measure. Let $\mathbb{E}|g(X)| < \infty$. Show that

$$\mathbb{E}[g(X)|Y = y] = \int_{\mathbb{R}} g(x) f_{X|Y}(x|y) dy$$

where

$$f_{X|Y}(x|y) = \begin{cases} \frac{p_{X,Y}(x,y)}{\int_{\mathbb{R}} p_{X,Y}(\tilde{x},y)d\tilde{x}}, & \text{if integral in denominator is not 0} \\ \mathbf{1}_{[0,1]}, & \text{in the opposite case} \end{cases}$$

is the conditional density.

6. Suppose that X_1 and X_2 are independent integrable r.v.'s. Prove that

$$\mathsf{P}\{X_1 + X_2 \in B | X_1\} = P_{X_2}(B - X_1),$$

where P_{X_2} is the distribution of X_2 and $B - x = \{y - x : \ y \in B\}$ is the translation of the set B.

7. Problems 2,3 from Section 9.2, p.333

8. Let $(X_k)_{k \in \mathbb{N}}$ be an i.i.d. sequence, each of the r.v.'s possessing a density $p(\cdot)$. Let

$$\begin{aligned} S_n &= X_1 + \ldots + X_n \\ X_n^* &= \max(X_1, \ldots, X_n) \\ S_n^* &= \max(S_1, \ldots, S_n) \end{aligned}$$

Are the following processes Markov? If yes find the transition probabilities and densities (if these exist):

   (a) X_n
   (b) X_n^*
   (c) S_n
   (d) S_n^*
   (e) (S_n, S_n^*)

9. Let $(X_n)_{n \geq 0}$ be a sequence of r.v.'s. and τ is a stopping time. Prove that τ and X_τ are $\mathcal{F}_{\leq \tau}$-measurable.

10. Let $\tau_1 \leq \tau_2$ be stopping times for a sequence $(X_n)_{n \geq 0}$. Prove that $\mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$.

11. Let τ_1 and τ_2 be stopping times for a sequence $(X_n)_{n \geq 0}$. Prove that $\tau_1 \wedge \tau_2, \tau_1 \vee \tau_2$ and $\tau_1 + \tau_2$ are also stopping times.

12. Suppose $(\mathcal{F}_n)_{n \geq 0}$ is a filtration and Y is a r.v. with $\mathbb{E}|Y| < \infty$. Prove that $X_n = \mathbb{E}(Y|\mathcal{F}_n)$ is an $(\mathcal{F}_n)$-martingale.

13. Let $(X_n)_{n \geq 0}$ be a homogeneous Markov process with transition probability $P(x, n, B)$ and h is a bounded function such that

$$\int_{\mathbb{R}} P(x, n, dy) h(y) \leq \lambda h(x)$$

for some $\lambda > 0$ and all x. Show that $(\lambda^{-n} h(X_n), \mathcal{F}_n)$ is a supermartingale, where $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$.