Homework assignments for Math 7244 Spring 2008

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1 Due on Friday Feb 8 2008

1. Let $(X_n)_{n\in\mathbb{N}}$ be an i.i.d. positive sequence, and $S_n = X_1 + \ldots + X_n$. Let $N_t = \sup\{n : S_n \le t\}$. Prove that $(N_t)_{t\in\mathbb{R}_+}$ is a stochastic process.
2. Let $(W_t)_{t\in\mathbb{R}_+}$ be a Wiener process. Find $\mathsf{cov}(W_s, W_t)$.
3. Prove that every Borel set in $\mathbb{R}^d$ is regular, i.e. for every probability measure μ, every $\varepsilon > 0$ there is a compact set K and open set U such that $K \subset B \subset U$ and $\mu(U \setminus K) < \varepsilon$.
4. Use characteristic functions to prove the existence of a Wiener process (up to continuity of paths)
5. Let $(X_t)_{t\in[0,1]}$ be an (uncountable) family of i.i.d. r.v.'s with nondegenerate distribution. Prove that there is no continuous modification.
6. Show that the proof of the Kolmogorov–Chentsov theorem actually implies that the constructed continuous modification of the original process satisfies the Hölder property.
7. A multidimensional version of the Kolmogorov–Chentsov theorem. Suppose $d \ge 1$, and there is a stochastic field $X : [0,1]^d \times \Omega \to \mathbb{R}$ that satisfies $\mathbb{E}|X(s) - X(t)|^\alpha \le C|s-t|^{d+\beta}$ for some $\alpha, \beta, C > 0$ and all $t, s \in [0,1]^d$. Prove that there is a continuous modification of X on $[0,1]^d$.

The following series of problems (culminating in problem 4) is optional. They won't be graded.

1. Let $\Omega = [0,1], \mathcal{F} = \mathcal{B}([0,1]), \mathsf{P} = \text{Leb}$. For each $\omega \in \Omega$ define $a_k(\omega)$ via

$$\omega = \sum_{k=1}^{\infty} a_k(\omega) 2^{-k},$$

where $a_k = 0$ for large k if $\omega = j/2^n$ for some j, n. Prove that (a_k) is an i.i.d. sequence with $\mathsf{P}\{a_k = 1\} = \mathsf{P}\{a_k = 0\} = 1/2$.

2. Prove that if (a_k) is an i.i.d. sequence with $\mathsf{P}\{a_k = 1\} = \mathsf{P}\{a_k = 0\} = 1/2$, then $U = \sum_{k=1}^{\infty} a_k 2^{-k}$ is uniformly distributed on $[0, 1]$.
3. Let F be a distribution function. Define $F^{-1}(y) = \inf\{x : F(x) > y\}$, $y \in [0, 1]$. Let $X = F^{-1}(U)$ where U is uniformly distributed on $[0, 1]$. Prove that $\mathsf{P}\{X \leq x\} = F(x)$ for all x.
4. Combine these problems to construct a sequence of i.i.d. real-valued random variables with given one-dimensional distribution on the probability space $([0, 1], \mathcal{B}([0, 1]), \mathrm{Leb})$.

2 Due on February 20 2008

1. Prove the following statement. Suppose there is a family of ch.f. $(\phi_{s,t}(\cdot))_{0\le s<t}$ such that for all $\lambda \in \mathbb{R}$ and all $t_1 < t_2 < t_3$,
$$\phi_{t_1,t_2}(\lambda)\phi_{t_2,t_3}(\lambda) = \phi_{t_1,t_3}(\lambda).$$
Then for every distribution function F there is a stochastic process $(X_t)_{t\in\mathbb{R}_+}$ with independent increments such that $X_0 \sim F$ and $\mathbb{E}e^{i\lambda(X_t-X_s)} = \phi_{s,t}(\lambda)$ for all $s < t$ and $\lambda \in \mathbb{R}$.

2. Show that the Kolmogorov–Chentsov theorem cannot be relaxed: inequality
$$\mathbb{E}|X_t - X_s|^\alpha \le C|t-s|$$
is not sufficient for existence of a continuous modification. Hint: consider the Poisson process.

3. Prove that there exists a Poisson process such that:
   (a) its realizations are nondecreasing, taking only whole values a.s.
   (b) its realizations are continuous on the right a.s.
   (c) all the jumps of the realizations are equal to 1 a.s.

4. Give an example of a non-Gaussian 2-dimensional random vector with Gaussian marginal distributions.

5. Let $Y \sim \mathcal{N}(a, C)$ be a d-dimensional random vector. Let $Z = AY$ where A is an $n \times d$ matrix. Prove that Z is Gaussian and find its mean and covariance matrix.

6. Prove that $(s,t) \mapsto e^{-|t-s|}$ is nonnegatively definite.

7. Prove that if X is a Gaussian vector in $\mathbb{R}^d$ with parameters (a, C) and C is non-degenerate, then the distribution of X is absolutely continuous w.r.t. Lebesgue measure and the density is
$$p_X(x) = \frac{1}{det(C)^{1/2}(2\pi)^{d/2}} e^{-\frac{1}{2}\langle C^{-1}(x-a),(x-a)\rangle}.$$

8. Find a condition on the mean $a(t)$ and covariance function $r(s,t)$ that guarantees existence of a continuous Gaussian process with these parameters.

9. Suppose $(X_0, X_1, \ldots, X_n)$ is a (not necessarily centered) Gaussian vector. Show that there are constants $c_0, c_1, \ldots, c_n$ such that

$$\mathbb{E}(X_0|X_1, \ldots, X_n) = c_0 + c_1 X_1 + \ldots + c_n X_n.$$

3 Due by April 7 2008

1. Let (x_n) be a sequence of points in a metric space E. Prove that δ_{x_n} converges weakly to δ_y iff $x_n \to y$.

2. Suppose (X_n) is a family of Gaussian processes in $C([0,1])$ for some $T > 0$. Find a condition ensuring convergence in distribution of these proceses in $C([0,1])$.

3. Let $S_n = \xi_1 + \ldots + \xi_n$ where (ξ_k) are i.i.d with $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}\xi_1^2 = 1$. Let $Z_n = S_1 + \ldots + S_n$. Use Donsker's Invariance Principle to find a sequence of numbers $b(n)$ such that the sequence $Z_n/b(n)$ has a non-trivial limit in distribution (i.e. a distribution that is not concentrated at one point), and prove the convergence.

4. Let $(X_n)_{n\in\mathbb{N}}$ be a family of continuous processes on $[0,1]$. Prove that the following two conditions are suuficient for tightness of distributions of (X_n) on $C([0,1])$:

   (a) There are constants $\gamma, M > 0$ such that $\mathbb{E}|X_n(0)|^\gamma < M$ for all n.

   (b) There are constants $C, \alpha, \beta > 0$ such that for all n, and any $t, s \in [0,1]$,
   $$\mathbb{E}|X(t) - X(s)|^\alpha \le C|t-s|^{1+\beta}.$$

   What has to be changed if one considers continuous processes defined on $[0,1]^d$ with arbitrary $d \in \mathbb{N}$?

5. Show that the function
   $$P(s,x,t,\Gamma) = \int_\Gamma \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}}\, dy$$
   is a Markov transition probability function for the standard Wiener process.

6. Prove that a process X is Markov iff
   $$\mathsf{P}\{AB|X(t)\} \overset{a.s.}{=} \mathsf{P}\{A|X(t)\}\mathsf{P}\{B|X(t)\}$$
   for any t, and any $A \in \sigma(X(s), s \le t)$, $B \in \sigma(X(s), s \ge t)$.

7. Prove that if τ_1 and τ_2 are stopping times then $\tau_1 \wedge \tau_2$ is a stopping time.

8. Let $W(t)$ be a Wiener process w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$. Let a and b be two positive numbers. Define

$$\begin{aligned} \tau_0 &= 0, \\ \tau_{2k-1} &= \inf\{t \geq \tau_{2k-2} : W(t) \geq a\}, \\ \tau_{2k} &= \inf\{t \geq \tau_{2k-1} : W(t) \leq -b\}, \end{aligned}$$

   for all $k \in \mathbb{N}$. Prove that τ_n is a stopping time w.r.t. $(\mathcal{F}_t)$ for all $n \in \mathbb{N}$.

9. Let $W(t)$ be a Wiener process w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $a > 0$, and let

$$\tau = \inf\{t : W(t) > a\}.$$

   Show that τ is a stopping time w.r.t $(\mathcal{F}_{t+})_{t\geq 0}$, where $\mathcal{F}_{t+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$.

10. Show that if $\tau_1 \leq \tau_2 \leq \ldots$ are stopping times w.r.t. to a filtration $(\mathcal{F}_t)$, then $\tau = \lim_{n\to\infty} \tau_n$ is also a stopping time w.r.t. to $(\mathcal{F}_t)$.

11. Let $\mathcal{F}_\tau = \{A : \ A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$ for a filtration $(\mathcal{F}_t)$ and a stopping time τ. Show that $\mathcal{F}_\tau$ is a σ-algebra.

12. Give an example of the following: τ is not a stopping time, $\mathcal{F}_\tau$ is not a σ-algebra.

13. Show that $\mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$ if $\tau_1 \leq \tau_2$ are stopping times.

14. Prove: If a process X_t is adapted to $(\mathcal{F}_t)$ and its all trajectories are right-continuous, then it is progressively measurable.

15. Prove that any process progressively measurable with respect to a filtration is adapted to that filtration.

4 Due on April 29

For all problems below, W, W_1, W_2 are assumed to be independent Wiener processes w.r.t. a filtration satisfying "the usual conditions". All other processes are assumed continuous and adapted.

1. Find

   $$\lim_{diam(t)\to 0} \sum_j A(t_j)(W_1(t_{j+1}) - W_1(t_j))(W_2(t_{j+1}) - W_2(t_j))$$

   in L^2 for a bounded process A.

   Comment. This problem may be used to prove a multi-dimensional Itô formula.

2. Let $a \in \mathbb{R}, \sigma > 0$. Use Itô's formula to prove that

   $$V(t) = V_0 e^{at} + \int_0^t e^{a(t-s)} \sigma dW(s)$$

   satisfies the Langevin equation

   $$dV(t) = aV(t)dt + \sigma dW(t), \quad V(0) = V_0$$

   Comment. Notice that in the absence of the noise ($\sigma = 0$) this would be a linear ODE. So, this is the linear dynamics perturbed by additive noise (i.e. coefficient in front of dW is constant)

3. Prove that if $f(s)$ is a determninstic function (i.e. it does not depend on $\omega \in \Omega$), then

   $$I(t) = \int_0^t f(s) dW(s)$$

   is a Gaussian process. Find its mean and covariance function. (Use Itô's isometry)

4. Suppose $X_n(t) = V(n+t),\ t \geq 0, n \in \mathbb{N}$ where V is given in Problem 2 above. Assuming $a < 0$, show that X_n converges in distribution in $C[0,T]$ for as $n \to \infty$ any $T > 0$. Find the limiting process.

5. Prove that

   $$X(t) = X_0 e^{(r-\sigma^2/2)t + \sigma W(t)}$$

   satisfies

   $$dX(t) = rX(t)dt + \sigma X(t) dW(t), \quad X(0) = X_0.$$

Comment. The stochastic differential XdW in the r.h.s. is called multiplicative noise.

6. Prove that a square-integrable martingale M has orthogonal increments in the sense of L^2. Use this property to derive that $\mathbb{E}M^2(t)$ is a non-decreasing function in t.

7. Let $(M_t, \mathcal{F}_t)$ be a continuous bounded martingale, and $\langle M\rangle_t$ be the compensator for M^2. Let $t_1 < t_2 \le t_3 < t_4$. Prove that

   $$\mathbb{E}((M_{t_2}-M_{t_1})^2-(\langle M\rangle_{t_2}-\langle M\rangle_{t_1}))((M_{t_4}-M_{t_3})^2-(\langle M\rangle_{t_4}-\langle M\rangle_{t_3})) = 0.$$

8. Prove that

   $$X(t) = (1-t)\int_0^t \frac{1}{1-s}dW(s)$$

   satisfies

   $$dX(t) = -\frac{X(t)}{1-t}dt + dW(t), \quad X(0) = 0.$$

   Compute the mean and covariance function of X.

9. Prove that with probability 1

   $$\lim_{t\uparrow 1} X(t) = 0,$$

   where X is the process from the previous problem.

   Comment. The process X is called the Brownian bridge (its graph connects points $(0,0)$ and $(1,0)$)