Homework assignments for Math 7244 Spring 2007

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1

1.1 List A, due on Thursday Feb 8 2007

1. Let $(X_n)_{n\in\mathbb{N}}$ be an i.i.d. positive sequence, and $S_n = X_1 + \ldots + X_n$. Let $N_t = \sup\{n : S_n \le t\}$. Prove that $(N_t)_{t\in\mathbb{R}_+}$ is a stochastic process.

2. Let $(W_t)_{t\in\mathbb{R}_+}$ be a Wiener process. Find $\mathsf{cov}(W_s, W_t)$.

3. Use characteristic functions to prove the existence of a Wiener process (up to continuity of paths)

4. Prove that Haar functions form a complete orthonormal system in $L^2([0,1], \mathcal{B}([0,1]), \mathrm{Leb})$.

5. Let $\Omega = [0,1], \mathcal{F} = \mathcal{B}([0,1]), \mathsf{P} = \mathrm{Leb}$. For each $\omega \in \Omega$ define $a_k(\omega)$ via

$$\omega = \sum_{k=1}^{\infty} a_k(\omega) 2^{-k},$$

where $a_k = 0$ for large k if $\omega = j/2^n$ for some j, n. Prove that (a_k) is an i.i.d. sequence with $\mathsf{P}\{a_k = 1\} = \mathsf{P}\{a_k = 0\} = 1/2$.

6. Show that the Kolmogorov–Chentsov theorem cannot be relaxed: inequality

$$\mathbb{E}|X_t - X_s|^\alpha \le C|t-s|$$

is not sufficient for existence of a continuous modification. Hint: consider the Poisson process.

7. Let $(X_t)_{t\in[0,1]}$ be an (uncountable) family of i.i.d. r.v.'s with nondenerate distribution. Prove that there is no continuous modification.

8. Prove that if (a_k) is an i.i.d. sequence with $\mathsf{P}\{a_k = 1\} = \mathsf{P}\{a_k = 0\} = 1/2$, then $U = \sum_{k=1}^{\infty} a_k 2^{-k}$ is uniformly distributed on $[0,1]$.

9. Let F be a distribution function. Define $F^{-1}(y) = \inf\{x : F(x) > y\}$, $y \in [0,1]$. Let $X = F^{-1}(U)$ where U is uniformly distributed on $[0,1]$. Prove that $\mathsf{P}\{X \le x\} = F(x)$ for all x.

10. Prove the following statement. Suppose there is a family of ch.f. $(\phi_{s,t}(\cdot))_{0 \le s < t}$ such that for all $\lambda \in \mathbb{R}$ and all $t_1 < t_2 < t_3$,

$$\phi_{t_1,t_2}(\lambda)\phi_{t_2,t_3}(\lambda) = \phi_{t_1,t_3}(\lambda).$$

    Then for every distribution function F there is a stochastic process $(X_t)_{t \in \mathbb{R}_+}$ with independent increments such that $X_0 \sim F$ and $\mathbb{E}e^{i\lambda(X_t - X_s)} = \phi_{s,t}(\lambda)$ for all $s < t$ and $\lambda \in \mathbb{R}$.

11. Let $Y \sim \mathcal{N}(a, C)$ be a d-dimensional random vector. Let $Z = AY$ where A is an $n \times d$ matrix. Prove that Z is Gaussian and find its mean and covariance matrix.

12. Prove that $(s,t) \mapsto \min(s,t)$ is nonnegatively definite. Hint: use functions $\mathbf{1}_{[0,t]} \in L^2[0,1]$.

1.2 List B, Other recommended problems

1. $\approx$ Problem 2.4 from Chapter 2. Prove that every Borel set in $\mathbb{R}^d$ is regular, i.e. for every probability measure μ, every $\varepsilon > 0$ there is a compact set K and open set U such that $K \subset B \subset U$ and $\mu(U \setminus K) < \varepsilon$. Notice the difference between this statement and the Problem in the book ("compact" instead of "closed").

2. Problem 2.9 from Chapter 2: continuous modifications for random fields.

3. Problem 3.2 from Chapter 1: explicit construction of Poisson process.

2

2.1 List A, due on February, 27

1. Prove that if X is a Gaussian vector in $\mathbb{R}^d$ with parameters (a, C) and C is non-degenerate, then the distribution of X is absolutely continuous w.r.t. Lebesgue measure and the density is
$$p_X(x) = \frac{1}{det(C)^{1/2}(2\pi)^{d/2}} e^{-\frac{1}{2}\langle C^{-1}(x-a),(x-a)\rangle}.$$
2. Find a condition on the mean $a(t)$ and covariance function $r(s,t)$ that guarantees existence of a continuous Gaussian process with these parameters.
3. Suppose $(X_0, X_1, \ldots, X_n)$ is a (not necessarily centered) Gaussian vector. Show that there are constants $c_0, c_1, \ldots, c_n$ such that
$$\mathbb{E}(X_0|X_1, \ldots, X_n) = c_0 + c_1X_1 + \ldots + c_nX_n.$$
4. Let (x_n) be a sequence of points in a metric space E. Prove that δ_{x_n} converges weakly to δ_y iff $x_n \to y$.
5. Suppose (X_n) is a family of Gaussian processes in $C([0,T])$ for some $T > 0$. Find a condition ensuring convergence in distribution of these proceses in $C([0,T])$.
6. Let $S_n = \xi_1 + \ldots + \xi_n$ where (ξ_k) are i.i.d with $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}\xi_1^2 = 1$. Let $Z_n = S_1 + \ldots + S_n$. Find a sequence of numbers $b(n)$ such that the sequence $Z_n/b(n)$ has a nontrivial limit in distribution (i.e. a distribution that is not concentrated at one point), and prove the convergence.
7. Consider a sequence of real-valued continuous random fields $X^{(m)}$ defined on $[0,1]^d$. What must be changed in the condition given in Problem 4.11 from Chapter 2 to guarantee tightness in $C([0,1]^d)$ for $d > 1$? A brief sketch of a proof is sufficient.
8. Problem 4.12 from Chapter 2.

2.2 List B, Other recommended problems

Chapter 2, problems 4.1, 4.11, 4.16.

3

3.1 List A, due on April, 12

1. Show that the function

$$P(s,x,t,\Gamma) = \int_\Gamma \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}}\,dy$$

   is a Markov transition probability function for the standard Wiener process.

2. Let $W(t)$ be a Wiener process w.r.t. a filtration $(\mathcal{F}_t)_{t\ge 0}$. Let a and b be two positive numbers. Define

$$\begin{aligned}\tau_0 &= 0,\\ \tau_{2k-1} &= \inf\{t \ge \tau_{2k-2} : W(t) \ge a\},\\ \tau_{2k} &= \inf\{t \ge \tau_{2k-1} : W(t) \le -b\},\end{aligned}$$

   for all $k \in \mathbb{N}$. Prove that τ_n is a stopping time w.r.t. $(\mathcal{F}_t)$ for all $n \in \mathbb{N}$.

3. Let $W(t)$ be a Wiener process w.r.t. a filtration $(\mathcal{F}_t)_{t\ge 0}$. Let $a > 0$, and let

$$\tau = \inf\{t : W(t) > 0\}.$$

   Show that τ is a stopping time w.r.t $(\mathcal{F}_{t+})_{t\ge 0}$, where $\mathcal{F}_{t+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$.

4. Show that if $\tau_1 \le \tau_2 \le \ldots$ are stopping times w.r.t. to a filtration $(\mathcal{F}_t)$, then $\tau = \lim_{n\to\infty} \tau_n$ is also a stopping time w.r.t. to $(\mathcal{F}_t)$.

5. Let $\mathcal{F}_\tau = \{A : \ A \cap \{\tau \le t\} \in \mathcal{F}_t\}$ for a filtration $(\mathcal{F}_t)$ and a stopping time τ. Show that $\mathcal{F}_\tau$ is a σ-algebra.

6. Give an example of the following: τ is not a stopping time, $\mathcal{F}_\tau$ is not a σ-algebra.

7. Show that $\mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$ if $\tau_1 \le \tau_2$ are stopping times.

8. Prove: If a process X_t is adapted to $(\mathcal{F}_t)$ and its all trajectories are right-continuous, then it is progressively measurable.

9. In the setting of the previous problem prove that X_τ is $\mathcal{F}_\tau$-measurable for any stopping time τ.

10. Let W_t be a Wiener process, and $\tau_b = \inf\{t > 0 : W(t) = b\}$ where $b \geq 0$. For each $b > 0$ find a constant $C(b)$ such that distributions of τ_b and $C(b)\tau_1$ coincide. (Use the Reflection Priciple to find the distribution of τ_b). *Due to the Strong Markov property, the process τ_b has independent increments as a function of b. That statement together with this problem implies that the distribution of τ_b is stable.*

3.2 List B, Other recommended problems

Read Section 2.9 of the book and solve all the problems from that section.

4 List A, due before the final exam

For all problems below, W, W_1, W_2 are assumed to be independent Wiener processes w.r.t. a filtration satisfying "the usual conditions". All other processes are assumed continuous and adapted.

1. Find

$$\lim_{diam(t)\to 0} \sum_j A(t_j)(W_1(t_{j+1}) - W_1(t_j))(W_2(t_{j+1}) - W_2(t_j))$$

   in L^2 for a bounded process A.

   Comment. This problem may be used to prove a multi-dimensional Itô formula.

2. Let $a \in \mathbb{R}, \sigma > 0$. Use Itô's formula to prove that

$$V(t) = V_0 e^{at} + \int_0^t e^{a(t-s)} \sigma dW(s)$$

   satisfies the Langevin equation

$$dV(t) = aV(t)dt + \sigma dW(t), \quad V(0) = V_0$$

   Comment. Notice that in the absence of the noise ($\sigma = 0$) this would be a linear ODE. So, this is the linear dynamics perturbed by additive noise (i.e. coefficient in front of dW is constant)

3. Prove that if $f(s)$ is a determninstic function (i.e. it does not depend on $\omega \in \Omega$), then

$$I(t) = \int_0^t f(s)dW(s)$$

   is a Gaussian process. Find its mean and covariance function. (Use Itô's isometry)

4. Suppose $X_n(t) = V(n+t),\ t \geq 0, n \in \mathbb{N}$ where V is given in Problem 2 above. Assuming $a < 0$, show that X_n converges in distribution in $C[0,T]$ for as $n \to \infty$ any $T > 0$. Find the limiting process.

5. Prove that

$$X(t) = X_0 e^{(r-\sigma^2/2)t + \sigma W(t)}$$

   satisfies

$$dX(t) = rX(t)dt + \sigma X(t)dW(t), \quad X(0) = X_0.$$

Comment. The stochastic differential XdW in the r.h.s. is called multiplicative noise.

6. Prove that a square-integrable martingale M has orthogonal increments in the sense of L^2. Use this property to derive that $\mathbb{E}M^2(t)$ is a non-decreasing function in t.

7. Let
$$I(t) = \int_0^t A(t)dW(t),$$
and τ is a stopping time with $\tau \leq T$ a.s. Prove that
$$I_\tau = \int_0^T A(t)\mathbf{1}_{[0,\tau]}(t)dW(t)$$

8. Prove that
$$X(t) = (1-t)\int_0^t \frac{1}{1-s}dW(s)$$
satisfies
$$dX(t) = -\frac{X(t)}{1-t}dt + dW(t), \quad X(0) = 0.$$
Compute the mean and covariance function of X.

9. Prove that with probability 1
$$\lim_{t\uparrow 1} X(t) = 0,$$
where X is the process from the previous problem.

Comment. The process X is called the Brownian bridge (its graph connects points $(0,0)$ and $(1,0)$)