Homework assignments for Math 7245 Fall 2007

Instructor: Yuri Bakhtin

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We assume that a filtered probability space with the "usual conditions" is given: $(\Omega, \mathcal{F}, (F_t)_{t\ge 0}, \mathsf{P})$. All the processes below are assumed to be progressively measurable.

1.1 List A, due on Tuesday Oct 2 2007

1. Suppose $M \in \mathcal{M}_2^c$, X is a simple process, and $I_t = \int_0^t X(s)dM(s)$. Prove that
$$\mathbb{E}[(I_t - I_s)^2|\mathcal{F}_s] = \mathbb{E}\left[\int_s^t X^2(r)d\langle M\rangle_r|\mathcal{F}_s\right]$$

2. Let $M \in \mathcal{M}_2^c$. Prove that
$$Y \cdot (X \cdot M) = (YX) \cdot M$$
for simple processes X, Y. Find reasonable conditions on X and Y guaranteeing the correctness of this identity in the sense of square integrable martingales. Extend this result to local martingales.

3. Show that
$$\inf_{t>0} R(t) = 0,$$
where R is the Bessel(2) process emitted from a positive point. (This can be done by modification of the proof of non-attainability of 0 given in class)

4. Let R be a Bessel(d) process, for $d \ge 3$. Use Itô's formula to find m such that $R^m(t)$ is a local martingale. Is it a martingale?

5. Let $N \in \mathcal{M}^{c,loc}$. Prove that boundedness of $\langle N\rangle_T$ implies that $N(t)$ and $N^2(t) - \langle N\rangle_t$, $t \le T$ are uniformly integrable.

6. Let $0 = A_0(t) + A_1(t)W(t)$ for all t, where A_0 and A_1 are C^1 processes, and $W(t)$ is a Wiener process. Prove that $A_0 \equiv 0$ and $A_1 \equiv 0$.

7. Let

$$0 = A_0(t) + A_1(t)W_1(t) + A_2(t)W_2(t) + A_{11}(t)W_1^2(t) + A_{12}(t)W_1W_2(t) + A_{22}(t)W_2^2(t),$$

   for all t, where W_1, W_2 are independent Wiener processes and all processes $A_.$ are C^1. Show that in fact all A-processes are identical zero. (Optional: prove a generalization of this statement for higher-order polynomials in several Wiener processes)

8. Problem 3.12.

9. Let $dY(t) = X(t)dt + dW(t)$. Assume that X is a bounded stochastic process. Let $K(t) = M(t)Y(t)$, where

$$M(t) = \exp\left\{-\int_0^t X(s)dW(s) - \frac{1}{2}\int_0^t X^2(s)ds\right\}.$$

   Use Itô's formula to prove that K is a martingale.

1.2 List B, Other recommended problems

1. Problems 3.14, 3.15, 4.7

2. Read Sections 3.2, 3.3, 3.4(A,B).

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2.1 List A, Due by Tuesday, Oct 23 2007

1. Problem 3.14
2. Problem 4.6
3. Problem 4.7
4. Let D be a bounded domain in $\mathbb{R}^d$, and $c, g : \bar{D} \to \mathbb{R}$ are continuous. Suppose $u \in C^2(\mathbb{R}^d)$, and

$$\begin{aligned} \frac{1}{2}\Delta u(x) + c(x)u(x) &= -g(x) \\ u(x) &= \phi(x), \end{aligned}$$

where $\phi : \partial D \to \mathbb{R}$ is a continuous function. Show that

$$\begin{aligned} u(x) = \mathbb{E} \Bigg[&\exp\left\{ \int_0^\tau c(x + W(s)) \right\} \phi(x + W(\tau)) \\ &+ \int_0^\tau \exp\left\{ \int_0^t c(x + W(s)) \right\} g(x + W(t)) dt \Bigg], \end{aligned}$$

where W is the standard Wiener process in $\mathbb{R}^d$, and τ is the ∂D-hitting time for $x + W$.

2.2 List B, Other recommended problems

Problem 2.25

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3.1 List A, Due by Friday, Dec 7 2007

1. Let $|b(t,x)|^2 + |\sigma(t,x)|^2 \leq K^2(1+|x|^2)$ for some $K > 0$ and all t, x. Consider Picard's iteration scheme:

$$X_0(t) \equiv x_0,$$
$$X_{k+1}(t) = x_0 + \int_0^t b(s, X_k(s))ds + \int_0^t \sigma(s, X_k(s))dW(s), \quad k \geq 0,$$

and prove that there are constants $\alpha, \beta > 0$ such that for all $k \in \mathbb{N}$ and all $t \geq 0$,

$$\mathbb{E}X_k^2(t) \leq \alpha e^{\beta t}.$$

2. Consider a Markov process X in $\mathbb{R}^2$ given by

$$X_1(t) = X_1(0) + W(t),$$
$$X_2(t) = X_2(0) + \int_0^t W(s)ds.$$

Find its generator on C^2-functions with compact support.

3. Let X solve the following SDE with Lipschitz coefficients:

$$X(t) = x_0 + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s)$$

with $x_0 \in [0,1]$. Let $\tau = \inf\{t : X(t) \notin [0,1]\}$.

Express $\mathsf{P}\{\tau > t\}$ as a solution to a PDE problem.

4. Suppose b, σ and g are bounded and Lipschitz and consider a function

$$v(t,x) = \mathbb{E}\int_0^t g(X(s))ds,$$

where

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s).$$

Prove that v is a solution of

$$\partial_t v(t,x) = Av(t,x) + g(x),$$
$$v(0,x) = 0.$$

where A is the infinitesimal operator associated with coefficients b, σ.

5. Consider the following SDE:

$$dX(t) = -bX(t)dt + \sigma dW(t),$$

   where b and σ are nonnegative constants.

   (a) Find an explicit solution to this equation.
   (b) Find the transition density.
   (c) Find the generator of the associated Markov semigroup.
   (d) Find the stationary density for this process.

6. Problem 1.2 from Chapter 5.

7. Exercise 2.28(i) from Chapter 5

8. Problem 3.13 from Chapter 5