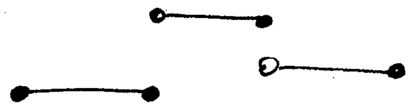
$$X(t) = X_t(\omega) = \xi_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=0}^{n-1} \xi_i(\omega)\, \mathbb{1}_{(t_i, t_{i+1}]} \qquad \left(\begin{array}{c}\text{simple}\\ \text{PM process}\end{array}\right)$$



ξ_i - $\mathcal{F}_{t_i}$ measurable.

$$\int_0^T X(t)\,dW(t) = \sum_{i=0}^{n-1} \xi_i \left[W(t_{i+1}) - W(t_i)\right]$$

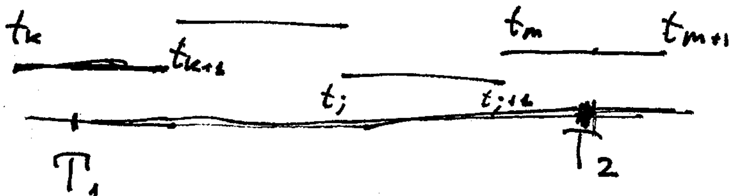
Properties

1) Continuous martingale as function of T

Continuity: obvious.

Martingale: have to check $E\left[\int_0^{T_2} X(t)\,dW(t) \,\middle|\, \mathcal{F}_{T_1}\right] = \int_0^{T_1} X(t)\,dW(t)$, a.s.

or, equivalently, $E\left[\int_{T_1}^{T_2} X(t)\,dW(t) \,\middle|\, \mathcal{F}_{T_1}\right] \overset{a.s.}{=} 0$



$$\int_{T_1}^{T_2} = \int_0^{T_2} - \int_0^{T_1}$$

$$E\left[\xi_k \left[W(t_{k+1}) - W(T_1)\right] \,\middle|\, \mathcal{F}_{T_1}\right] = \xi_k \underbrace{E\left[W(t_{k+1}) - W(T_1) \,\middle|\, \mathcal{F}_{T_1}\right]}_{=0}$$

$k < j < m$:

$$E\left[\xi_j \left(W(t_{j+1}) - W(t_j)\right) \,\middle|\, \mathcal{F}_{T_1}\right] = E\left[\xi_j \underbrace{E\left(W(t_{j+1}) - W(t_j) \,\middle|\, \mathcal{F}_{t_j}\right)}_{=0} \,\middle|\, \mathcal{F}_{T_1}\right] \overset{a.s.}{=} 0$$

$E\left[\xi_m W(T_2) - W(t_m) \,\middle|\, \mathcal{F}_{T_1}\right] = 0$ analogously.

2) $E\left(\int_0^T X(t)\,dW(t)\right)^2 = E\int_0^T X^2(t)\,dt$

This follows from the next property:

3) $E\left[\left(\int_{T_1}^{T_2} X(t)\,dW(t)\right)^2 \Big| \mathcal{F}_{T_1}\right] = E\left[\int_{T_1}^{T_2} X^2(t)\,dt \Big| \mathcal{F}_{T_1}\right]$

$$\left(\int_{T_1}^{T_2} X(t)\,dW(t)\right)^2 = \sum_{ij} \xi_i \xi_j \big(\underbrace{W(t_{i+1}) - W(t_i)}_{\Delta W_i}\big)\big(\underbrace{W(t_{j+1}) - W(t_j)}_{\Delta W_j}\big)$$

where for brevity I assume $t_0 = T_1$, $t_n = T_2$

If $i \neq j$, say $i < j$, then

$$E\left(\xi_i \xi_j\, \Delta W_i \cdot \Delta W_j \mid \mathcal{F}_{T_1}\right) = E\left(\xi_i \xi_j\, \Delta W_i\, E(\Delta W_j \mid \mathcal{F}_{t_j}) \mid \mathcal{F}_{T_1}\right) = 0$$

If $i = j$, then

$$E(\xi_i \xi_j\, \Delta W_i \Delta W_j \mid \mathcal{F}_{T_1}) = E\left[\xi_i^2 (\Delta W_i)^2 \mid \mathcal{F}_{T_1}\right]$$

$$= E\left(\xi_i^2\, E\left[(\Delta W_i)^2 \mid \mathcal{F}_{t_i}\right] \Big| \mathcal{F}_{T_1}\right]$$

$$= E\left[\xi_i^2\, E(\Delta W_i)^2 \mid \mathcal{F}_{T_1}\right]$$

$$= E\left[\xi_i^2 (t_{i+1} - t_i) \mid \mathcal{F}_{T_1}\right]$$

$$= E\left[\int_{t_i}^{t_{i+1}} X^2(t)\,dt \Big| \mathcal{F}_{T_1}\right].$$

4) $\int_0^T \left(\alpha X(t) + \beta Y(t)\right) dW(t) = \alpha \int_0^T X(t)\,dW(t) + \beta \int_0^T Y(t)\,dW(t)$

<u>Thm</u> For every bounded PM X there is a sequence $X^{(m)}$ of simple processes with $\lim_{m\to\infty} E\int_0^T |X_t^{(m)} - X_t|^2 dt = 0$ $(*)$

<u>Proof</u> 1) Let X be continuous & adapted

$$X_t^{(m)}(\omega) := X_0(\omega)\,\mathbb{1}_{\{0\}}(t) + \sum_{k=0}^{2^m-1} X_{\frac{kT}{2^m}}(\omega)\,\mathbb{1}_{\left(\frac{kT}{2^m}, \frac{(k+1)T}{2^m}\right]}(t)$$

For each ω, t $X_t^{(m)}(\omega) \longrightarrow X_t(\omega)$

Dominated convergence $\Rightarrow (*)$

2) X: PM

$$\tilde{X}_t^{(m)} = m\int_{(t-\frac{1}{m})\vee 0}^{t} X_s\, ds \qquad \text{continuous \& adapted}$$

for each ω $\tilde{X}_t^{(m)}(\omega) \to X_t(\omega)$ for Lebesgue a.e. t (Thm of Lebesgue)

Therefore & Dominated convergence implies

$$\lim_{m\to\infty} E\int_0^t |\tilde{X}_t^{(m)} - X_t^2|\, dt = 0 \qquad (**)$$

Now, each $\tilde{X}_t^{(m)}$ can be approximated by simple processes due to 1).

Together with $(**)$, this implies our claim