In all the following problems

$$S_n = X_1 + X_2 + \ldots + X_n,$$

where $(X_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. r.v.'s with

$$\mathsf{P}\{X_1 = 1\} = \mathsf{P}\{X_1 = -1\} = \frac{1}{2}.$$

1. Prove that

   $$\mathsf{P}\{S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n = 0\} = \frac{1}{4}\mathsf{P}\{S_1 \geq 0, S_2 > 0, \ldots, S_{n-3} \geq 0, S_{n-2} = 0\}.$$

2. Let N_n be the number of returns of S_n to 0 up to time n:

   $$N_n = \sum_{k=1}^{n} H_k,$$

   where

   $$H_k = \begin{cases} 1, & S_k = 0, \\ 0, & S_k \neq 0. \end{cases}$$

   Prove that $\mathbb{E}N_n$ grows as $\sqrt{n}$ as $n \to \infty$. Hint: start with taking expectations of both sides of the above formula for N_n and finding $\mathbb{E}H_k$.

   Comment: The result is interesting because it shows that the number of returns to 0 grows sublinearly (grows slower for large n). This supports our vision of the random walk as a sequence of lengthy excursions away from 0.

3. (This is a follow-up to one of the homework problems) Let g_{2n} be the last time of visit of S_k to 0 before time $2n$. Compare the probabilities: $\mathsf{P}\{g_{2n} > n\}$ and $\mathsf{P}\{g_{2n} < n\}$.

   Comment: the result is sort of counterintuitive, but it also supports the general picture of random walk that has been discussed in class.

4. Let r and k be two numbers with $r > k$ and $r > 0$. For any $n > 0$, find

   $$\mathsf{P}\left\{\max_{k=1,\ldots,n} S_k \geq r, S_n = k\right\}$$

   and

   $$\mathsf{P}\left\{\max_{k=1,\ldots,n} S_k \leq r, S_n = k\right\}.$$