

Kuratowski's "14" theorem

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```
In[1]:= SetDirectory["1:"]; << goedel74.23a; << tools.m

:Package Title: goedel74.23a          2005 October 23 at 12:00 noon

It is now: 2005 Oct 25 at 19:46

Loading Simplification Rules

TOOLS.M                      Revised 2005 October 19

weightlimit = 40
```

summary

Kuratowski showed that at most 14 sets can be constructed from a given subset of a topological space by repeated applications of interior, closure and complementation. The method used to derive this fact is here generalized to arbitrary **core** and **hull** constructors.

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In[2]:= "J. L. Kelley, General Topology, Van
        Nostrand, 1955. Page 45, footnote, and page 57, Problem E.;"
```

derivation

Theorem 1.

```
In[3]:= SubstTest[implies, subclass[u, v],
              subclass[hull[x, u], hull[x, v]], {u -> core[y, hull[x, z]], v -> hull[x, z]}]
```

```
Out[3]= subclass[hull[x, core[y, hull[x, z]]], hull[x, z]] == True
```

```
In[4]:= subclass[hull[x_, core[y_, hull[x_, z_]]], hull[x_, z_]] := True
```

Theorem 2.

```
In[5]:= SubstTest[implies, subclass[u, v],
              subclass[core[x, u], core[x, v]], {u -> core[x, z], v -> hull[y, core[x, z]}]
```

```
Out[5]= subclass[core[x, z], core[x, hull[y, core[x, z]]] == True
```

```
In[6]:= subclass[core[x_, z_], core[x_, hull[y_, core[x_, z_]]] := True
```

Lemma.

```
In[7]:= SubstTest[implies, subclass[u, v], subclass[core[x, u], core[x, v]],
  {u -> hull[y, core[x, hull[y, z]]], v -> hull[y, z]}]
Out[7]= subclass[core[x, hull[y, core[x, hull[y, z]]]], core[x, hull[y, z]]] = True
In[8]:= (% /. {x -> x_, y -> y_, z -> z_}) /. Equal -> SetDelayed
```

Theorem 3.

```
In[9]:= SubstTest[and, subclass[u, v], subclass[v, u],
  {u -> core[x, hull[y, z]], v -> core[x, hull[y, core[x, hull[y, z]]]}]
Out[9]= True == equal[core[x, hull[y, z]], core[x, hull[y, core[x, hull[y, z]]]]
In[10]:= core[x_, hull[y_, core[x_, hull[y_, z_]]] := core[x, hull[y, z]]
```

Lemma.

```
In[11]:= SubstTest[implies, subclass[u, v], subclass[hull[x, u], hull[x, v]],
  {u -> core[y, z], v -> core[y, hull[x, core[y, z]]}]
Out[11]= subclass[hull[x, core[y, z]], hull[x, core[y, hull[x, core[y, z]]]] = True
In[12]:= (% /. {x -> x_, y -> y_, z -> z_}) /. Equal -> SetDelayed
```

Theorem 4.

```
In[13]:= SubstTest[and, subclass[u, v], subclass[v, u],
  {u -> hull[x, core[y, z]], v -> hull[x, core[y, hull[x, core[y, z]]]}]
Out[13]= True == equal[hull[x, core[y, z]], hull[x, core[y, hull[x, core[y, z]]]]
In[14]:= hull[x_, core[y_, hull[x_, core[y_, z_]]] := hull[x, core[y, z]]
```