Tutorial #2. Getting by with fewer primitives.

Johan G. F. Belinfante
2001 February 11

```
<< goedel52.a51; << tests.m

:Package Title: GOEDEL52.A51    2001 February 8 at 10:00 p.m.

It is now:  2001 Feb 11 at 14:27

Loading Simplification Rules

TESTS.M         Revised 2000 December 30

weightlimit = 30


Context switch to ‘Goedel‘Private is needed for ReplaceTest

Just ignore the error message about Unterminated use of BeginPackage

Get::bebal : Unterminated uses of BeginPackage or Begin in << tests.m.
```

■ Introduction

Goedel uses **V, E, pairset, complement, intersection, cart, domain, inverse, flip** and **rotate** as primitives. One can eliminate **inverse** by defining it in terms of **flip**. This still leaves 9 primitives.

```
domain[flip[cart[x, V]]]

inverse[x]
```

Alternatively, one can retain **inverse**, and instead eliminate **flip**. Bernays shows that one can reduce the number of primitives in class theory to the following seven: **E, complement, intersection, cart, domain, inverse**, and **rotate**. Actually, Bernays does not use **rotate**, but instead uses a slightly different primitive, which is closely related:

```
class[pair[pair[u, v], w], member[pair[u, pair[v, w]], x]]

rotate[inverse[x]]
```

The following reference to the work of Bernays is readily accessible.

Paul Bernays, "Axiomatic Set Theory," North Holland Publishing Co, 1968.
Reprinted by Dover Publications, Inc., 1991. See page 64.

The ideas undoubtedly go back much further. On page 7 of his monograph on the consistency of the axiom of choice and the generalized continuum hypothesis, Goedel refers to a paper published by Bernays in 1937. Goedel's comment is that by adding an axiom asserting the existence of the class **range[SINGLETON]** of all singletons, the two axioms for **flip** and **rotate** can be replaced by a single axiom.

■ Constructions based on the Bernays primitives.

The Dover book reference only comments briefly on how the constructions should proceed, but I did not have much difficulty filling in the gaps. What we do below is to show how to construct Goedel's primitives from those of Bernays. This is best done in stages. Among the easiest constructions are those for **range** and **union**:

```
domain[inverse[x]]
```

```
range[x]
```

```
complement[intersection[complement[x], complement[y]]]
```

```
union[x, y]
```

■ Eliminating V as primitive.

Bernays notes that one does not need to take **V** as a primitive because it can be expressed in terms of **E**.

```
union[E, complement[E]]
```

```
V
```

Alternatively, one could introduce the universal class **V** as

```
domain[E]
```

```
V
```

■ Composite in terms of the Bernays primitives.

Bernays gives a formula for **composite** in terms of the primitives, which can be simplified as follows:

```
inverse[domain[rotate[intersection[cart[y, V],
rotate[cart[x, V]]]]]]
```

```
composite[x, y]
```

Once one has **composite** available, it is easy to construct the subset relation **S** and the identity relation **Id**.

```
inverse[complement[composite[E, complement[inverse[E]]]]]
```

```
S
```

```
intersection[S, inverse[S]]
```

```
Id
```

Alternately, one could define the identity relation **Id** more directly as

```
inverse[complement[composite[inverse[E], complement[E]]]]
```

```
Id
```

The function **SINGLETON** can now also be defined, which seems to obviate the need for a separate axiom concerning the existence of **range[SINGLETON]**.

```
intersection[E, composite[complement[composite[E, complement[Id]]]]]
```

```
SINGLETON
```

■ Defining image, singleton, and related formulas.

The restriction of the identity relation is defined as:

```
intersection[Id, cart[x, x]]
```

```
id[x]
```

Next one can construct **image**, and use it to define the sum class **U[x]**, the power class **P[x]**, the unary intersection **A[x]**, and the singleton:

```
range[composite[x, id[y]]]
```

```
image[x, y]
```

```
image[inverse[E], x]
```

```
U[x]
```

```
complement[image[E, complement[x]]]
```

```
P[x]
```

```
complement[image[complement[inverse[E]], x]]
```

```
A[x]
```

```
intersection[P[x], complement[image[complement[E], x]]]
```

```
singleton[x]
```

The construction of **pairset[x,y]** follows trivially from this:

```
assert[equal[pairset[x, y], union[singleton[x], singleton[y]]]]
```

```
True
```

■ FIRST, SECOND, SWAP, flip[x] and cross[x,y]

The construction of **FIRST** and **SECOND** are given by:

```
rotate[cart[Id, V]]
```

```
SECOND
```

```
rotate[SECOND]
```

```
FIRST
```

Next one can construct **SWAP**, **flip** and **cross** in terms of the primitives:

```
intersection[composite[inverse[FIRST], SECOND], composite[inverse[SECOND], FIRST]]
```

```
SWAP
```

```
flip[x]
```

```
composite[x, SWAP]
```

```
intersection[composite[inverse[FIRST], x, FIRST], composite[inverse[SECOND], y, SECOND]]
```

```
cross[x, y]
```