Analiticity of the Lyapunov exponent of perturbed toral automorphisms.

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Abstract

We consider a dynamical system generated by a perturbation A_ε of an analytic Anosov diffeomorphism A_0 of $\mathbb{T}^d$. We show that, if A_0 admit a decomposition of $T\mathbb{T}^d$ in k invariant subspaces, such a decomposition can be extended in an analytic way to A_ε. This implies that the Lyapunov exponents, if non degenerate, are analytic functions of the perturbation.

1 Introduction

Hyperbolicity is one the main ingredient in the modern study of long time behavior of dynamical systems. Lyapunov exponents characterize in a quantitative way the hyperbolic nature of a dynamical system. The knowledge of the Lyapunov exponents, gives further insight on the dynamical properties of the system. However, notwithstanding their importance, in practice it is very hard to obtain rigorous quantitative information on Lyapunov exponents (see [13, 17] for interesting examples). Most known results originate from numerical simulations based on the classical algorithm introduced in [1]. Even the question of their regularity in the parameters defining a particular family of systems is largely outside analytical reach (see e.g. fig. 12 in [4]). It is thus already interesting to obtain results on very simple systems, e.g. perturbations of analytic diffeomorphisms of the torus in d dimension. In this respect the present work is an application of the techniques developed in [2, 8] to the problem studied in [18]. In this introduction we provide two examples where we hope our results will turn out to be a starting point for further research.

Another important quantity that plays a role in the analysis of chaotic dynamical systems is the fractal dimension of their attractor. There are several different definitions of fractal dimension (see e.g. [20]) but a widely accepted conjecture, called the Kaplan-Yorke (KY) conjecture, asserts that the Hausdorff dimension $\dim_{\mathrm{HD}}(\mu_A)$ of the SRB measure μ_A of an hyperbolic dynamical system A on a d-dimensional manifold is equal to its Lyapunov dimension $\dim_{\mathrm{L}}(\mu_A)$ introduced in [10, 6].

In the case of a small perturbation of a system that preserves the Lebesgue measure the Lyapunov dimension is

$$\dim_{\mathrm{L}}(\mu_A) = d - \frac{\sigma}{\lambda_1} \tag{1.1}$$

where λ_1 is the smallest Lyapunov exponent while σ is the average phase space contraction rate and it is equal to the sum of all Lyapunov exponents. The KY conjectures was proved in [20] for systems in 2 dimension and in [11] for random dynamical systems. On the other hand, it is easy to see that the KY conjecture cannot be true in general. Indeed we can consider the direct product of two different independent Anosov systems on $\mathbb{T}^2$ for each of which the result in [20] apply; in this case the Hausdorff dimension of the SRB measure of the system, seen as a diffeomorphism of $\mathbb{T}^4$, is the sum of the dimensions of the two components, and a direct computation shows that this is in general different from the r.h.s. of (1.1), see Section 5 for more details. Clearly a direct product is not a generic system. However it would be enough to show that the Hausdorff dimension of the SRB measure is continuous under perturbations to show that the KY conjecture in generically false in an open neighbourhood of the direct product of two independent system. A somehow similar argument is developed in [9]. For more details on this example, see the discussion in section 5, where we formulate a slightly different conjecture.

When considering dynamical systems in very high dimension d it is natural to parametrize the Lyapunov exponents λ_i by $x = i/d$, with $0 \leq x \leq 1$. In simulations of realistic physical system, like the FPU system (see fig. 2.2 in [12]) or the truncated Navier-Stokes equation (see fig. 5-7 in [7]) or the SLLOD particle system (see fig. 2 in [5]), it was observed that the resulting function $\lambda(x)$ approaches, when $d \to \infty$, a well defined limit that is given by a smooth function of x. A related property was proven for a system of particles, but with a somehow different definition of the Lyapunov exponents, in [19].

Another interesting property, clearly visible in the figures in [7] and [5], is that Lyapunov exponents tend to pair, that is $\lambda(x) + \lambda(1-x)$ is a constant independent of x. This property was rigorously proven for a simple model of non-equilibrium statistical mechanics in [3], and linked to the transport properties of the system in [5]. The significance of such a property for the Navier-Stokes system is unclear but surely of interest. We believe that both phenomena, convergence to a smooth spectrum and pairing, can be analysed rigorously in a chain of coupled Anosov system like that studied in [2, 8] with a suitable extension of the results in this work.

2 Definitions and results

We consider an analytic diffeomorphism $A_0 : \mathbb{T}^d \mapsto \mathbb{T}^d$ and assume that it defines an Anosov dynamical system. This means that:

- there exists $\psi \in \mathbb{T}^d$ such that the set $\{A_0^m(\psi) : \ m \in \mathbb{Z}\}$ is dense;
- the tangent space $\mathrm{T}_\psi \mathbb{T}^d$ at $\psi \in \mathbb{T}^d$ can be written as the direct sum of two subspaces $W_0^u(\psi)$, $W_0^s(\psi)$, with $\dim W_0^{s,u}(\psi) = d^{s,u}$, invariant under the action of the differential DA_0 of A_0, that is

$$DA_0(\psi) W_0^{u,s}(\psi) = W_0^{u,s}(A_0(\psi));$$

- $W_0^{s,u}(\psi)$ depends continuously on ψ;
- the splitting $W_0^{s,u}(\psi)$ is hyperbolic, that is there exist constants $\Theta > 0$ and $0 < \lambda < 1$ such that for every $n \geq 0$

$$\begin{aligned} \|DA_0^n \mathbf{w}\| &\leq \Theta\lambda^n \|\mathbf{w}\|, \qquad \mathbf{w} \in W_0^s(\psi) \\ \|DA_0^{-n} \mathbf{w}\| &\leq \Theta\lambda^n \|\mathbf{w}\|, \qquad \mathbf{w} \in W_0^u(\psi). \end{aligned} \tag{2.2}$$

To construct a perturbed system, we consider the lifting $\widetilde{A}_0$ of A_0 to a map from $\mathbb{T}^d$ to its universal covering $\mathbb{R}^d$ and an analytic function $F \in C^\omega(\mathbb{T}^d, \mathbb{R}^d)$. We can then define

$$A_\varepsilon(\psi) := \widetilde{A}_0(\psi) + \varepsilon F(\psi) \qquad \bmod 2\pi, \tag{2.3}$$

where ϵ is a small parameter and $\bmod 2\pi$ represents the standard projection from $\mathbb{R}^d$ to $\mathbb{T}^d$.

Given $f : \mathbb{T}^d \longrightarrow \mathbb{R}^k$ and $0 < \beta < 1$, we define the *Hölder seminorm* of f as

$$|f|_\beta := \sup_{\psi,\psi' \in \mathbb{T}^d} \frac{\|f(\psi) - f(\psi')\|}{\delta(\psi, \psi')^\beta}.$$

where $\delta(\psi, \psi')$ is the metric inherited by $\mathbb{T}^d$ from the euclidean metric on $\mathbb{R}^d$. Moreover we set

$$\|f\|_\beta := \|f\|_\infty + |f|_\beta$$

where, as usual, $\|f\|_\infty := \sup_{\psi \in \mathbb{T}^d} \|f(\psi)\|$. If $\|f\|_\beta$ is finite we say that f is (β-)*Hölder continuous*.

From structural stability it follows that, for ε small enough, the dynamical system $(\mathbb{T}^d, A_\varepsilon)$ is still Anosov. More precisely, it follows from [2] that, still for ε small enough, there exists a *conjugation* H_ε between the *perturbed* system $(\mathbb{T}^d, A_\varepsilon)$ and the *unperturbed* one $(\mathbb{T}^d, A_0)$, that is a solution of the equation

$$H_\varepsilon \circ A_0 = A_\varepsilon \circ H_\varepsilon.$$

Moreover H_ε is an homeomorphism, Hölder continuous in ψ and analytic in ε that can be written as

$$H_\varepsilon(\psi) = \psi + \sum_{n \geq 1} \varepsilon^n h^{(n)}(\psi) \tag{2.4}$$

with $\|h^{(n)}\|_\beta \leq C_\beta^n$ for every $n > 1$ and any β small enough, with $C_\beta > 0$ depending on β.

We further assume that the splitting (2.2) of $\mathrm{T}\mathbb{T}^d$ can be refined. More precisely we assume that for every $\psi \in \mathbb{T}^d$ there exist k subspaces $V_1(\psi), \ldots, V_k(\psi)$ of $\mathrm{T}_\psi\mathbb{T}^d$, with $\dim(V_i) = d_i$, such that $\mathrm{T}_\psi\mathbb{T}^d = \oplus_{i=1}^k V_i(\psi)$ and

$$DA_0(\psi)V_i(\psi) = V_i(A_0(\psi)) \quad \text{for} \quad i = 1, \ldots, k. \tag{2.5}$$

Moreover we assume that the V_i are β–Hölder continiuous for some $\beta > 0$.

It is natural to ask whether (2.5) extends to A_ε, i.e. if we can find solutions to the equation

$$DA_\varepsilon(\psi)W_{i,\varepsilon}(\psi) = W_{i,\varepsilon}(A_\varepsilon(\psi)), \qquad i = 1, \ldots, k, \tag{2.6}$$

with $W_{i,0}(\psi) = V_i(\psi)$. For this to happen we must require the presence of gaps between the contraction/expansion rates of DA_0 restricted on the V_i. To make this precise, for $i = 1, \ldots, k$, we consider the projector $P_i(\psi) : \mathrm{T}_\psi \mathbb{T}^d \to V_i(\psi)$ associated with the decomposition $(V_j)_{j=1}^k$ and, setting $\mathcal{L}_i(\psi) = P_i(A_0(\psi))DA_0(\psi)\big|_{V_i(\psi)}$, we assume that there exists N_0 such that for $n > N_0$ we have

$$\|\mathcal{L}_i^{[n]}(\psi)\|\|(\mathcal{L}_{i+1}^{[n]}(\psi))^{-1}\| \leq 1, \qquad i = 1, \ldots, k-1 \tag{2.7}$$

where $\mathcal{L}_i^{[n]}(\psi) = \prod_{i=0}^{n-1} \mathcal{L}_i(A_0^i(\psi))$.

Notwithstanding this, notice that a solution to (2.6), if existing at all, can hardly be analytic in ε since A_ε depends on ε and $W_{i,\varepsilon}$ is, in general, only Hölder continuous in ψ. For this reason we pull back the subspaces $W_{i,\varepsilon}$ to the unperturbed d-torus and we look for linear maps $\mathcal{V}_{i,\varepsilon}(\psi) : V_i(\psi) \mapsto \mathrm{T}_\psi \mathbb{T}^d$ and $\mathcal{L}_{i,\varepsilon}(\psi) : V_i(\psi) \mapsto V_i(A_0\psi)$ that satisfy

$$DA_\varepsilon(H_\varepsilon(\psi))\mathcal{V}_{i,\varepsilon}(\psi) = \mathcal{V}_{i,\varepsilon}(A_0(\psi))\mathcal{L}_{i,\varepsilon}(\psi). \tag{2.8}$$

Observe indeed that the subspaces $W_{i,\varepsilon}(\psi) := \mathcal{V}_{i,\varepsilon}(H_\varepsilon^{-1}(\psi))V_i(H_\varepsilon^{-1}(\psi))$ satisfy (2.6)

We can now formulate our main theorem whose proof is in Section 3 and 4.

Theorem 2.1. *There exists $\bar{\beta} > 0$ and $\bar{\varepsilon}(\beta) > 0$ such that, for $\beta < \bar{\beta}$ and $\varepsilon < \bar{\varepsilon}(\beta)$ there exist invertible linear maps $\mathcal{V}_\varepsilon(\psi) : \mathrm{T}_\psi\mathbb{T}^d \mapsto \mathrm{T}_\psi\mathbb{T}^d$ and $\mathcal{L}_\varepsilon(\psi) : \mathrm{T}_\psi\mathbb{T}^d \mapsto \mathrm{T}_{A_0(\psi)}\mathbb{T}^d$, analytic in ε and β-Hölder continuous in ψ such that, calling*

$$\mathcal{L}_{\varepsilon,i,j}(\psi) := P_i(\psi)\mathcal{L}_\varepsilon(\psi)\big|_{V_j} \quad \mathcal{V}_{\varepsilon,i,j}(\psi) := P_i(\psi)\mathcal{V}_\varepsilon(\psi)\big|_{V_j},$$

we have

$$\mathcal{L}_{\varepsilon,i,j}(\psi) = 0 \qquad \text{and} \qquad \mathcal{V}_{0,i,j}(\psi) = 0 \qquad \textit{for } i \neq j \text{ and } \psi \in \mathbb{T}^d$$

and

$$DA_\varepsilon(H_\varepsilon(\psi))\mathcal{V}_\varepsilon(\psi) = \mathcal{V}_\varepsilon(A_0(\psi))\mathcal{L}_\varepsilon(\psi)\,. \tag{2.9}$$

A particularly interesting case is when all V_i have dimension 1, that is when $k = d$ so that $d_i = 1$ for $i = 1, \ldots, d$. In this case we can write $V_i(\psi) = \mathrm{span}(\mathbf{v}_i(\psi))$ where $\mathbf{v}_i(\psi)$ are d vector that form a basis of $\mathrm{T}_\psi\mathbb{T}^d$ for every $\psi \in \mathbb{T}^d$ and satisfy

$$DA_0(\psi)\mathbf{v}_i(\psi) = \Lambda_i(\psi)\mathbf{v}_i(A_0\psi)$$

with $\Lambda_i(\psi)$ β_0–Hölder continuous satisfying

$$\|\Lambda_i^{[n]}(\psi)\|_\infty\|(\Lambda_{i+1}^{[n]}(\psi))^{-1}\|_\infty \leq 1 \tag{2.10}$$

for $i = 1, \ldots, d-1$. From Theorem 2.1 we immediately get the following.

Theorem 2.2. *There exists $\bar{\beta} > 0$ and $\bar{\varepsilon}(\beta) > 0$ such that, for $\beta < \bar{\beta}$ and $\varepsilon < \bar{\varepsilon}(\beta)$ there exist d linearly independent vector fields $\mathbf{v}_{i,\varepsilon}(\psi)$ and d scalar functions $L_{i,\varepsilon}(\psi)$, analytic in ε, β-Hölder continuous in ψ, such that*

$$DA_\varepsilon(H_\varepsilon(\psi))\mathbf{v}_{i,\varepsilon}(\psi) = L_{i,\varepsilon}(\psi)\mathbf{v}_{i,\varepsilon}(A_0\psi) \tag{2.11}$$

where $\mathbf{v}_{i,0}(\psi) = \mathbf{v}_i(\psi)$, $L_{i,0}(\psi) = \Lambda_i(\psi)$.

From (2.11) it follows that setting $\mathbf{w}_{i,\varepsilon}(\psi) = \mathbf{v}_{i,\varepsilon}(H_\varepsilon^{-1}(\psi))$ and $\Lambda_{i,\varepsilon}(\psi) = L_{i,\varepsilon}(H_\varepsilon^{-1}(\psi))$ we get

$$DA_\varepsilon(\psi)\mathbf{w}_{i,\varepsilon}(\psi) = \Lambda_{i,\varepsilon}(\psi)\mathbf{w}_{i,\varepsilon}(A_\varepsilon(\psi)). \tag{2.12}$$

We can thus use (2.11) to study the Lyapunov exponent of A_ε.

Let μ_0 be the normalized volume measure on $\mathbb{T}^d$ and consider the *SRB* measure of A_ε defined as

$$\mu_\varepsilon := \lim_{n\to\infty} A_\varepsilon^n \mu_0.$$

where the limit is intended in the weak sense. Since A_ε is Anosov, μ_ε exists and is ergodic. Observe that

$$DA_\varepsilon^n := \prod_{k=0}^{n-1} DA_\varepsilon(A_\varepsilon^k)$$

so that the limit

$$\lambda_{i,\varepsilon}(\psi) := \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{\|DA_\varepsilon^n(\psi)\mathbf{w}_{i,\varepsilon}(\psi)\|}{\|\mathbf{w}_{i,\varepsilon}(\psi)\|}\right), \tag{2.13}$$

exists and it is μ_ε-a.e. constant. This implies that defining i-th *Lyapunov exponent* of A_ε as

$$\lambda_{i,\varepsilon} := \int_{\mathbb{T}^d} \log(|\Lambda_{i,\varepsilon}(\psi)|) d\mu_\varepsilon(\psi)$$

we have $\lambda_{i,\varepsilon}(\psi) = \lambda_{i,\varepsilon}$, μ_ε-a.e.

From the construction of the SRB measure in [8], it follows that if $f : \mathbb{T}^d \longrightarrow \mathbb{R}$ is an Hölder continuous function, then

$$\int_{\mathbb{T}^d} f \circ H_\varepsilon^{-1}(\psi) d\mu_\varepsilon(\psi)$$

is analytic in ε for ε small enough. From Theorem 2.2 we get the following corollary.

Corollary 2.3. *The i-th Lyapunov exponent can be written as*

$$\lambda_{i,\varepsilon} = \int_{\mathbb{T}^d} \log\left(|L_{i,\varepsilon}(H_\varepsilon^{-1}(\psi))|\right) d\mu_\varepsilon(\psi) \tag{2.14}$$

and it is an analytic function of ε for $\varepsilon < \bar{\varepsilon}(0)$.

The rest of the paper is organized as follows. Section 3 contains all details of the proof of Theorem 2.2 while Section 4 gives a description of the main modifications needed to obtain a proof of Theorem 2.1. Finally in Section 5 we report some final considerations and possibilities for further research.

3 Proof of Corollary 2.2

This section is devoted to the proof of Theorem 2.2. We first show that the coefficients of the power series expansion in ε of the solutions $\mathbf{v}_{i,\varepsilon}$ and $L_{i,\varepsilon}$ of (2.11) can be computed thanks to recursive relations. We then use these recursive relations to show that the power series thus obtained are convergent.

3.1 Perturbative construction for the solutions of (2.11)

Without loss of generality, we can assume that (2.10) holds with $N_0 = 1$. Indeed this can always be achieved by replacing A_0 with A_0^N.

We thus look for solutions of (2.11) as power series in ε, that is we write

$$\begin{aligned} \mathbf{v}_{i,\varepsilon}(\psi) &= \mathbf{v}_i(\psi) + \sum_{n\geq 1} \varepsilon^n \mathbf{v}_i^{(n)}(\psi), \\ L_{i,\varepsilon}(\psi) &= \Lambda_i(\psi) + \sum_{n\geq 1} \varepsilon^n L_i^{(n)}(\psi). \end{aligned} \tag{3.15}$$

Observe that, if $\mathbf{v}(\psi)$ and $L(\psi)$ are solutions of (2.11), then, for any $g(\psi) > 0$, also $\tilde{\mathbf{v}}(\psi) := g(\psi)\mathbf{v}(\psi)$ and $\tilde{L}(\psi) := \frac{g(\psi)}{g(A_0\psi)} L(\psi)$ are. To solve this ambiguity we first assume that $\|\mathbf{v}_i(\psi)\| = 1$. Then we consider the basis of $\mathrm{T}_\psi \mathbb{T}^d$ formed by the vector $\mathbf{u}_1(\psi), ..., \mathbf{u}_d(\psi)$ such that $\mathbf{u}_k(\psi)\cdot\mathbf{v}_i(\psi) = 1$ if $i = k$ and 0 otherwise so that any vector $\mathbf{f} \in \mathrm{T}_\psi \mathbb{T}^d$ we can write $\mathbf{f} = \sum_{k=1}^d f_k \mathbf{v}_k(\psi)$ with $f_k = \mathbf{u}_k(\psi) \cdot \mathbf{f}$. Observe moreover that the $\mathbf{u}_i$ satisfy

$$DA_0(A_0(\psi))^* \mathbf{u}_i(\psi) = \Lambda_i(\psi)\mathbf{u}_i(\psi) \tag{3.16}$$

where $DA_0(\psi)^*$ is the adjoint of $DA_0(\psi)$. We can now require that the component of $\mathbf{v}_{i,\varepsilon}$ along $\mathbf{v}_i$ be constant. More precisely, we require that $\mathbf{u}_i(\psi) \cdot \mathbf{v}_{i,\varepsilon}(\psi) = 1$ so that

$$\mathbf{u}_i(\psi) \cdot \mathbf{v}_i^{(n)}(\psi) = 0 \tag{3.17}$$

for every $n \geq 1$ and $i = 1, ..., d$.

We can now plug (3.15) in (2.11), expand both sides in power of ε and equate the resulting terms of the same order. The zero order equation is trivially satisfied while the first order of (2.11) reads

$$DA_0(\psi)\mathbf{v}_i^{(1)}(\psi) + DF(\psi)\mathbf{v}_i(\psi) = \Lambda_i(\psi)\mathbf{v}_i^{(1)}(A_0(\psi)) + L_i^{(1)}(\psi)\mathbf{v}_i(\psi), \tag{3.18}$$

where DF is the differential of F. Note that in (3.18) both $L_i^{(1)}(\psi)$ and $\mathbf{v}_i^{(1)}(\psi)$ are unknown.

Multiplying both sides by $\mathbf{u}_i(A_0(\psi))$ and using (3.16) and (3.17) we obtain

$$L_i^{(1)}(\psi) = D_{\mathbf{v}_i} F_i(\psi), \tag{3.19}$$

where $D_{\mathbf{v}_j} F(\psi) := DF(\psi)\mathbf{v}_j(\psi)$ while $D_{\mathbf{v}_i} F_k(\psi) := \mathbf{u}_k(A_0(\psi)) \cdot DF(\psi) \cdot \mathbf{v}_i$. On the other hand, multiplying (3.18) by $\mathbf{u}_k(A_0(\psi))$, $k \neq i$, we get

$$\Lambda_k(\psi)v_{ik}^{(1)}(\psi) + D_{\mathbf{v}_i} F_k(\psi) = \Lambda_i(\psi)v_{ik}^{(1)}(A_0\psi), \tag{3.20}$$

To simplify the notation, we set $\psi_m := A_0^m \psi$, $\mathbb{Z}^+ := \mathbb{N} \cup \{0\}$, $\mathbb{Z}^- := -\mathbb{N}$ and $\omega_{ik} := +$ if $i < k$ and $\omega_{ik} := -$ otherwise. The solution of (3.20) can now be written as

$$v_{ik}^{(1)}(\psi) = -\sum_{m\in\mathbb{Z}^{\omega_{ik}}} D_{\mathbf{v}_i} F_k(\psi_m) \frac{\Lambda_i^{[m]}(A_0(\psi))}{\Lambda_k^{[m+1]}(\psi)} \tag{3.21}$$

where $\Lambda_i^{[m]}(\psi)$ is defined after (2.7). The series in (3.21) is convergent due to our definition of $\mathbb{Z}^\omega$ and the ordering of the Λ_i in (2.10).

Thus with (3.19) and (3.21) we an have explicit expressions for the first order coefficients in (3.15). To obtain expressions for the higher order coefficients we write

$$D_{\mathbf{v}_i} F_k(H_\varepsilon(\psi)) = \sum_{r\geq 0} \varepsilon^n \Phi_{ik}^{(n)}(\psi)\,.$$

A direct computation shows that, for $n \geq 1$,

$$\Phi_{ik}^{(n)}(\psi) = \sum_{s=1}^{n} \frac{1}{s!} \sum_{\substack{q_1,\ldots,q_s \\ q_1+\ldots+q_s=n \\ \sigma_1,\ldots,\sigma_s\in\{1,\ldots,d\}}} \mathbf{u}_k(A_0(\psi)) \cdot \left[\prod_{l=1}^{s} D_{\mathbf{v}_{\sigma_l}} D_{\mathbf{v}_i} F(\psi)\right] \prod_{l=1}^{s} h_{\sigma_l}^{(q_l)}(\psi)$$

while $\Phi_{ik}^{(0)}(\psi) = D_{\mathbf{v}_i} F_k(\psi)$. Setting

$$\eta_{ik}^{(n)}(\psi) := \sum_{j=1}^{d} \sum_{p=0}^{n-1} v_{ij}^{(p)}(\psi) \Phi_{jk}^{(n-1-p)}(\psi), \tag{3.22}$$

and multiplying by $\mathbf{u}_k(A_0(\psi))$, $k \neq i$, the n-th order of (2.11), we get

$$\Lambda_k(\psi) v_{ik}^{(n)}(\psi) + \eta_{ik}^{(n)}(\psi) = \Lambda_i(\psi) v_{ik}^{(n)}(A_0\psi) + \sum_{p=1}^{n} L_i^{(p)}(\psi) v_{ik}^{(n-p)}(A_0\psi). \tag{3.23}$$

Analogously to (3.21) we can write the k-th component of $\mathbf{v}_i^{(n)}(\psi)$ as

$$v_{ik}^{(n)}(\psi) = \sum_{m\in\mathbb{Z}^{\omega ik}} \sum_{p=1}^{n-1} \left(L_i^{(p)}(\psi_m) v_{ik}^{(n-p)}(\psi_{m+1}) - \eta_{ik}^{(n)}(\psi_m) \right) \frac{\Lambda_i^{[m]}(A_0(\psi))}{\Lambda_k^{[m+1]}(\psi)} \tag{3.24}$$

Finally, multiplying the n-th order of (2.11) by $\mathbf{u}_i(A_0(\psi))$, we get

$$L_i^{(n)}(\psi) = \eta_{ii}^{(n)}(\psi)$$

which reads

$$L_i^{(n)}(\psi) = \sum_{j=1}^{d} \sum_{p=0}^{n-1} v_{ij}^{(p)}(\psi) \Phi_{ji}^{(n-1-p)}(\psi). \tag{3.25}$$

Observe that the r.h.s. of (3.22) and (3.24) depends only on $\mathbf{v}_i^{(p)}$ with $p < n$. In this way (3.22) and (3.24) provide a recursive construction of the coefficient of the power series expansion (3.15). We will show in the following section that these series converge and thus provide a solutions for (2.11).

3.2 Proof of convergence of the perturbative series

Before studying the behavior of the solutions (3.22) and (3.24) we state and prove some technical Lemmas.

Lemma 3.4. *For β small enough there exists a constant $B_\beta > 0$ such that*

$$\|\Phi_{ik}^{(n)}\|_\beta \le B_\beta^{n+1},$$

for all $n \ge 0$.

Proof. Since $F(\psi)$ is analytic, there exists a constant $R > 1$ such that for all $s \ge 1$ and $\sigma_1, \dots, \sigma_s \in \{1, ..., d\}$ we have

$$\Big\| \prod_{l=1}^{s} D_{\mathbf{v}_{\sigma_l}} F(\psi) \Big\|_\infty \le s! R^s \,. \tag{3.26}$$

Moreover observe that for any f and g we have

$$|fg|_\beta \le \|f\|_\infty |g|_\beta + |f|_\beta \|g\|_\infty, \qquad \|fg\|_\beta \le \|f\|_\beta \, \|g\|_\beta \,. \tag{3.27}$$

The statement is clearly true for $n = 0$. For $n \ge 1$, from the bound in (2.4) using (3.26), and (3.27) we get

$$\begin{aligned}
\|\Phi_{ik}^{(n)}\|_\infty \le & \sum_{s=1}^{n} \frac{1}{s!} \sum_{\substack{q_1,\dots,q_s \ge 1 \\ q_1+\dots+q_s = n \\ \sigma_1,\dots,\sigma_s \in \{1,\dots,d\}}} \Big\| \prod_{l=1}^{s} h_{\sigma_l}^{(q_l)} \Big\|_\beta \Big\| D_{\mathbf{v}_i} \prod_{l=1}^{s} D_{\mathbf{v}_{\sigma_l}} F_k \Big\|_\beta \\
\le & \sum_{s=1}^{n} \frac{1}{s!} \sum_{\substack{q_1,\dots,q_s \ge 1 \\ q_1+\dots+q_s = n \\ \sigma_1,\dots,\sigma_s \in \{1,\dots,d\}}} (s+1)! R^{s+1} C_\beta^n \\
\le & C_\beta^n \left(\sum_{s=1}^{n} (s+1) R^{s+1} \sum_{\substack{q_1,\dots,q_s \\ q_1+\dots+q_s = n}} 1 \right).
\end{aligned}$$

Observe now that

$$\sum_{\substack{q_1,\dots,q_s \\ q_1+\dots+q_s = n \ge 1}} 1 = \binom{n-1}{s-1} \le 2^{n-1}$$

so that

$$\|\Phi_{ik}^{(n)}\|_\beta \le (2C_\beta)^n \left(\sum_{s=1}^{n} (s+1) R^{s+1} \right) \le (n+1)^2 (2RC_\beta)^{n+1}.$$

□

Let us consider the sequence $\alpha = (\alpha_n)_{n=0}^{\infty}$ recursively defined as

$$\begin{cases} \alpha_0 := 1 \\ \alpha_n := \sum_{j=1}^{n-1} \alpha_{n-j}\alpha_j + \sum_{j=0}^{n-1} \alpha_j \, . \end{cases} \tag{3.28}$$

The recursive definition (3.28) is meant to mimic the basic structure of (3.24). We will first show that α grows at most geometrically with n and then use this fact to control the growth of $\|v_{ik}^{(n)}\|_\beta$.

Lemma 3.5. *Let α be defined as in* (3.28)*, then we have:*

$$\alpha_n \leq 4^n.$$

Proof. Given a sequence $a = (a_n)_{n=0}^{\infty}$ define the sequence Δa as

$$(\Delta a)_n := a_{n+1} - a_n$$

so that, for every sequences a, b, we have

$$\sum_{j=k}^{n-1} a_j \Delta b_j = a_n b_n - a_k b_k - \sum_{j=k}^{n} b_{j+1} \Delta a_j. \tag{3.29}$$

Using the operator Δ on α defined as in (3.28) we get the following difference equation

$$\Delta\alpha_n = \sum_{j=1}^{n-1} \alpha_j \Delta\alpha_{n-j} + 2\alpha_n, \tag{3.30}$$

and using (3.29) we get

$$\sum_{j=1}^{n-1} \alpha_j \Delta\alpha_{n-j} = \alpha_{n-1}\Delta\alpha_1 - \Delta\alpha_{n-1} - \sum_{j=1}^{n-1} \alpha_{j+1}\Delta\alpha_{n-j}.$$

On the other hand we have $\alpha_j \leq \alpha_{j+1}$ and $\Delta\alpha_1 = 2$, so that

$$\sum_{j=1}^{n-1} \alpha_j \Delta\alpha_{n-j} \leq 3\alpha_{n-1} - \alpha_n - \sum_{j=1}^{n-1} \alpha_j \Delta\alpha_{n-j},$$

from which

$$\sum_{j=1}^{n-1} \alpha_j \Delta\alpha_{n-j} \leq \frac{3}{2}\alpha_{n-1} - \frac{1}{2}\alpha_n. \tag{3.31}$$

Using (3.31) into (3.30) and the monotony of α_n we get

$$\Delta\alpha_n \leq \frac{3}{2}(\alpha_{n-1} + \alpha_n) \leq 3\alpha_n$$

and the thesis follows. □

Observe that the bound in (3.37) has a structure similar to the recursive definition (3.28). We will use this fact to bound $\|v_{ik}^{(n)}\|_\beta$. Letting $\Omega = \max\{\|DA_0\|_\infty, \|DA_0^{-1}\|_\infty^{-1}\}$, for any f and any $m \in \mathbb{Z}$ we get

$$\|f \circ A_0^m\|_\beta \leq \Omega^{\beta|m|}\|f\|_\beta. \tag{3.32}$$

This leads to the following Lemma.

Lemma 3.6. *For β small enough there exists a constant $E_\beta > 0$ such that*

$$\|v_{ik}^{(n)}\|_\beta \leq \alpha_n E_\beta^n, \qquad \|L_{ik}^{(n)}\|_\beta \leq \alpha_n E_\beta^n \tag{3.33}$$

for all $n \geq 0$.

Proof. From (3.32) and first of (3.27) it follows that

$$\|\Lambda_i^{[m]}\|_\beta \leq \frac{\Omega^{\beta|m|}}{|\Omega^\beta - 1|}\|\Lambda_i\|_\beta\|\Lambda_i\|_\infty^{m-1} \qquad \|(\Lambda_i^{[m]})^{-1}\|_\beta \leq \frac{\Omega^{\beta|m|}}{|\Omega^\beta - 1|}\|\Lambda_i\|_\beta\|(\Lambda_i)^{-1}\|_\infty^{m+1}$$

Using (3.24) and (3.32) we get

$$\begin{aligned}\|v_{ik}^{(n)}\|_\beta \leq & K_\beta \sum_{m\in\mathbb{Z}^{\omega_{ik}}} \sum_{j=1}^{n-1} \left(\left\|L_i^{(j)} \circ A_0^m \; v_{ik}^{(n-j)} \circ A_0^{m+1}\right\|_\beta + \|\eta_{ik}^{(n)} \circ A_0^m\|_\beta\right) \left(\Omega^{2\beta}\|\Lambda_i\|_\infty\|(\Lambda_k)^{-1}\|_\infty\right)^m \\ \leq & K_\beta \sum_{m\in\mathbb{Z}^{\omega_{ik}}} \left(\|\Lambda_i\|_\infty\|(\Lambda_k)^{-1}\|_\infty\right)^m \Omega^{3\beta(|m|+1)} \sum_{j=1}^{n-1}\left(\left\|L_i^{(j)} v_{ik}^{(n-j)}\right\|_\beta + \|\eta_{ik}^{(n)}\|_\beta\right)\end{aligned} \tag{3.34}$$

where $K_\beta := \|\Lambda_i\|_\beta\|\Lambda_k\|_\beta\|\Lambda_k\|_\infty^2/|\Omega^\beta - 1|^2$ while (3.22) and (3.25), together with Lemma 3.4, give

$$\|L_i^{(n)}\|_\beta\,,\ \|\eta_{ik}^{(n)}\|_\beta \leq \sum_{p=0}^{n-1}\sum_{l=1}^{d} \|v_{il}^{(p)}\|_\beta B_\beta^{n-p}. \tag{3.35}$$

Moreover observe that for β small enough we have

$$K_\beta\Omega \sum_{m\in\mathbb{Z}^{\omega^{ik}}} \left(\|\Lambda_i\|_\infty\|(\Lambda_k)^{-1}\|_\infty\right)^m \Omega^{3\beta|m|} = \frac{K_\beta\Omega\|\Lambda_i\|_\infty\|(\Lambda_k)^{-1}\|_\infty}{\omega_{ik}(1 - \|\Lambda_i\|_\infty\|(\Lambda_k)^{-1}\|_\infty\Omega^{3\omega_{ik}\beta})} := K_{1,\beta}. \tag{3.36}$$

Combining (3.34), (3.35) and (3.36) we get

$$\|v_{ik}^{(n)}\|_\beta \leq K_{1,\beta}\left(\sum_{j=1}^{n-1}\sum_{p=0}^{j-1}\sum_{l=1}^{d} \|v_{il}^{(p)}\|_\beta\|v_{ik}^{(n-j)}\|_\beta B_\beta^{j-p} + \sum_{j=0}^{n-1}\|v_{il}^{(j)}\|_\beta B_\beta^{n-j}\right) \tag{3.37}$$

We can now prove that

$$\|v_{ik}^{(n)}\|_\beta \leq \alpha_n E_\beta^n. \tag{3.38}$$

with $E_\beta = K_{1,\beta} B_\beta d$. It is easy to see that (3.38) holds for $n = 1$. Assuming (3.38) to be true for all $k \le n-1$ we plug it in (3.37) and we obtain

$$\begin{aligned}\|v_{ik}^{(n)}\|_\infty \le & dK_{1,\beta} B_\beta E_\beta^{n-1} \Big(\sum_{j=1}^{n-1} \alpha_{n-j} \sum_{p=0}^{j-1} \alpha_p + \sum_{j=0}^{n-1} \alpha_j \Big) \\ \le & E_\beta^n \Big(\sum_{j=1}^{n-1} \alpha_{n-j} \sum_{p=0}^{j-1} \alpha_p + \sum_{j=0}^{n-1} \alpha_j \Big) \\ \le & E_\beta^n \Big(\sum_{j=1}^{n-1} \alpha_{n-j} \alpha_j + \sum_{j=0}^{n-1} \alpha_j \Big) = E_\beta^n \alpha_n,\end{aligned}$$

where the last inequality follows from the fact that

$$\sum_{p=0}^{j-1} \alpha_p \le \alpha_j. \tag{3.39}$$

The second bound in (3.33) now follows from (3.35) and (3.39).

□

Combining Lemmas 3.5 and 3.6 we get that the series (3.15) converge for $|\varepsilon| < E_\beta^{-1}$ and thus define β-Hölder continuous solutions for (2.11).

Finally choosing $\bar{\varepsilon}(\beta) := (4dE_\beta)^{-1}$ we see that, for $|\varepsilon| \le \bar{\varepsilon}(\beta)$, Lemma 3.6, Lemma 3.5 and (3.15) imply that $\|v_{ik,\varepsilon}\|_\infty \le d^{-1}$ so that for every $\psi \in \mathbb{T}^d$ and every $i = 1, \ldots, d$ we have

$$\sum_{k \ne i} |v_{ik,\varepsilon}(\psi)| < 1,$$

while, by definition, $v_{ii,\varepsilon}(\psi) = 1$. Therefore, by Gershgorin circle theorem, the matrix $V_\varepsilon(\psi) := (v_{ik,\varepsilon}(\psi))_{1 \le i,k \le d}$ is invertible, i.e. the vector fields $\mathbf{v}_{i,\varepsilon}$ are linearly independent for every ψ. This conclude the proof of Theorem 2.1.

4 Proof of Theorem 2.1

Proceeding as in (3.15) we look for $\mathcal{V}_\varepsilon, \mathcal{L}_\varepsilon$ of the form

$$\begin{aligned}\mathcal{V}_\varepsilon(\psi) &= \mathcal{V}_0 + \sum_{n \ge 1} \varepsilon^n \mathcal{V}^{(n)}(\psi), \\ \mathcal{L}_\varepsilon(\psi) &= \mathcal{L}_0 + \sum_{n \ge 1} \varepsilon^n \mathcal{L}^{(n)}(\psi).\end{aligned} \tag{4.40}$$

where we can take $\mathcal{V}_0 = \mathrm{Id}$ and $\mathcal{L}_0 = A_0$. Plugging (4.40) into (2.9) and extracting the first order terms we get

$$A_{0,i} \mathcal{V}_{i,j}^{(1)}(\psi) - \mathcal{V}_{i,j}^{(1)}(A_0 \psi) A_{0,j} = -DF_{i,j}(\psi) + \mathcal{L}_{i,j}^{(1)}(\psi) \tag{4.41}$$

where $DF_{i,j}(\psi) := P_i DF(\psi)\big|_{V_j}$. Again we need to resolve the ambiguity of these equations so that we fix $\mathcal{V}^{(n)}_{i,i}(\psi) = 0$ for every $n \geq 1$ and $\psi \in \mathbb{T}^2$. Then we can solve (4.41) by setting

$$\mathcal{L}^{(1)}_{i,i}(\psi) = DF_{i,i}(\psi)$$

for $i = j$, while for $i < j$ we get

$$\mathcal{V}^{(1)}_{i,j}(\psi) = -\sum_{m=0}^{\infty} A_{0,i}^{-(m+1)} DF_{i,j}(A_0^m \psi) A_{0,j}^m \tag{4.42}$$

while an analogous expression holds for $i > j$, see (3.21). It is clear that, thanks to (2.7), the series (4.42) is convergent. We can now repeat the analysis in section 3 to show that (2.9) admit a solution analytic in ε and Hölder continuous in ψ.

5 Conclusion

In this section we present further details on a possible application of the present work by discussing a concrete and very simple example obtained by coupling two linear automorphisms B_1 and B_2 of $\mathbb{T}^2$. More precisely consider the system A_ε acting on $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ and given by (2.3) with $A_0(\psi) = A_0\psi \mod 2\pi$ and

$$A_0 = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}. \tag{5.43}$$

Assume moreover that, for $(\psi_1, \psi_2) \in \mathbb{T}^2 \times \mathbb{T}^2$, we have

$$F(\psi_1, \psi_2) = G_1(\psi_1) + G_2(\psi_2). \tag{5.44}$$

If $\lambda_{i,1,\varepsilon}$ and $\lambda_{i,2,\varepsilon}$, $i = 1, 2$, are the Lyapunov exponent of the dynamical system $B_i + \varepsilon G_i \mod 2\pi$ on $\mathbb{T}^2$ then, from [20], we know that the Hausdorff dimension of the SRB measure μ_ε is

$$\dim_{\mathrm{HD}}(\mu_\varepsilon) = 4 - \frac{\lambda_{1,1,\varepsilon} + \lambda_{1,2,\varepsilon}}{\lambda_{1,1,\varepsilon}} - \frac{\lambda_{2,1,\varepsilon} + \lambda_{2,2,\varepsilon}}{\lambda_{2,1,\varepsilon}} \tag{5.45}$$

while the KY conjecture gives

$$\dim_{\mathrm{L}}(\mu_\varepsilon) = 4 - \frac{\lambda_{1,1,\varepsilon} + \lambda_{1,2,\varepsilon} + \lambda_{2,1,\varepsilon} + \lambda_{2,2,\varepsilon}}{\lambda_{1,1,\varepsilon}} \tag{5.46}$$

From the present works it follows that if $B_1 \neq B_2$, so that $\lambda_{1,i,0} \neq \lambda_{2,i,0}$, then, for ϵ small, (5.45) and (5.46) give different results and the KY conjecture cannot be true. As stated in the introduction, it would suffice to show that $\dim_{\mathrm{HD}}(\mu_\varepsilon)$ is continuous in ε to obtain that the KY conjecture is not verified in an open neighbor of A_0. We think we can prove that $\dim_{\mathrm{HD}}(\mu_\varepsilon)$ is actually an analytic function of ε.

It is interesting to notice that if $B_1 = B_2$ and $G_1 = G_2$ then (5.45) and (5.46) agree. This suggests that, at least in this context, the KY conjecture can depend on some extra symmetry of the system. A natural conjecture is that the KY conjecture holds true when $B_1 = B_2$ and

$$F(\psi_1, \psi_2) = F(\psi_2, \psi_1) .$$

Notwithstanding this, if F is not of the form (5.44) while $B_1 = B_2$, our results only tell us that $\lambda_{1,1,\varepsilon} + \lambda_{2,1,\varepsilon}$ is a smooth function of ε. Thus a first step toward a proof of our conjecture is to extend the analysis in this paper to understand when and how a perturbation can resolve the degeneracy of the Lyapunov exponents.

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