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Absolute continuity of projected SRB measures of coupled Arnold cat map lattices

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Abstract. We study a d-dimensional coupled map lattice consisting of hyperbolic toral automorphisms (Arnold cat maps) that are weakly coupled by an analytic map. We construct the Sinai–Ruelle–Bowen measure for this system and study its marginals on the tori. We prove that they are absolutely continuous with respect to Lebesgue measure if and only if the coupling satisfies a non-degeneracy condition.

1. *Introduction*

There has been much interest recently in time-invariant measures of physical systems evolving under certain types of non-Hamiltonian deterministic dynamics. These dynamics are chosen (invented) with the intent of making these measures model the behavior of stationary non-equilibrium states of real physical systems: e.g. the 'Gaussian thermostated' dynamics [**EM**]. An interesting example is provided by the Moran and Hoover model of electric-current-carrying systems [**MH**]. A particle moves on a torus among fixed obstacles under the influence of an external electric field E and a *thermostat* which keeps the energy fixed (it would otherwise grow indefinitely). A very striking (initially surprising) result of the numerical simulations was that the stationary phase space density in a Poincaré section looked very 'fractal', i.e. singular with respect to the reference Lebesgue measure. The singular nature of the invariant measure was later proven rigorously (under suitable hypotheses) for $E \neq 0$, at least when E is small [**CELS**]. Further computer simulations and rigorous results strongly suggest that thermostated stationary measures are indeed generically singular with respect to Lebesgue measure [**R99**]. They correspond to the Sinai–Ruelle–Bowen (SRB) measures for these systems [**G95**].

The singular nature of these non-equilibrium measures, is a consequence of the modeling by deterministic dynamics. Alternative modeling of non-equilibrium systems, using some stochasticity, yields stationary measures which are absolutely continuous with respect to Lebesgue measure [**BLR**]. This has raised some questions about the consistency of these different modelings of such stationary non-equilibrium states [**HHP, H, Le, BDLR**]. Fortunately the answer is that for systems containing many particles the different models can yield the same physical behavior. The reason for this is that relevant observable properties of macroscopic physical systems correspond to sums of functions which depend only on the coordinates and velocities of one or a few particles, e.g. the electrical current is a sum over the velocities of individual particles. Their steady-state values can therefore be computed from the reduced one- or two-particle distribution functions and we expect these induced measures to be absolutely continuous with respect to Lebesgue measure even when the measure in the full phase space of the system is singular. It is the purpose of this paper to show, by explicit example, that this is indeed to be expected generically.

To do this we consider the reduced distributions or induced measure for a very idealized dynamical system made up of an infinite collection of Arnold cat maps of the two torus, indexed by a d-dimensional lattice. This dynamical system has typically an invariant measure which is singular with respect to Lebesgue measure. We prove, however, that under general conditions, the projected measure on a single torus is absolutely continuous with respect to Lebesgue measure. Note that our result is for a projection on an explicitly given surface on which the measure is singular in the absence of coupling to other systems—not just for a 'typical' projection. This requires some conditions on the interaction which we specify—those excluded are very special. They are essentially uncoupled systems.

2. *Definitions and results*

The dynamical systems that we consider in this paper are coupled map lattices [**PS**]. The phase space of such a system is given by a Cartesian product over a d-dimensional lattice $\Omega = \mathbb{Z}^d$ of finite-dimensional manifolds $\mathcal{N}$. In our case, $\mathcal{N}$ is the two-dimensional torus $\mathcal{N} = \mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ and the full phase space is $\mathcal{T} = \mathbb{T}^\Omega$, equipped with the product topology. We will construct the systems via finite-dimensional approximations, letting $\mathcal{T}_N = \mathbb{T}^{\Omega_N}$ where $\Omega_N = (\mathbb{Z}_N)^d$ and $\mathbb{Z}_N$ consists of integers of absolute value strictly less than N.

The dynamics in a coupled map lattice is defined by first fixing a dynamical system on each separate $\mathcal{N}$ and then coupling them appropriately. In the case at hand, let $A : \mathbb{T} \to \mathbb{T}$ be the Anosov dynamical system defined by the linear transformation $A \in GL_2(\mathbb{Z})$ with $|\det A| = 1$. We can define the *uncoupled* map $A : \mathcal{T} \to \mathcal{T}$ and respectively on $\mathcal{T}_N$ by letting A act on each copy of $\mathbb{T}$. More precisely, denoting by $\psi \in \mathbb{T}$ the points on the two-dimensional torus and by $\Psi = (\Psi_\mathbf{i})_{\mathbf{i}\in\Omega}$ those on $\mathcal{T}$, we set $(A\Psi)_\mathbf{i} = A\Psi_\mathbf{i}$. The use of the same symbol for the map on $\mathbb{T}$ and for the uncoupled map on $\mathcal{T}$ is done to avoid too cumbersome notations when no confusion can arise. The Lebesgue measures $d\psi$ on each torus and their product $d\Psi$, are invariant under A.

To describe the coupled map, let $f : \mathcal{T} \to \mathbb{R}^2$ be a map and define $\mathcal{A} : \mathcal{T} \to \mathcal{T}$ by

$$(\mathcal{A}\Psi)_{\mathbf{i}} = A\Psi_{\mathbf{i}} + \mathcal{F}_{\mathbf{i}}(\Psi), \quad \mathbf{i} \in \Omega \tag{2.1}$$

where

$$\mathcal{F}_{\mathbf{i}}(\Psi) = f(\tau_{-\mathbf{i}}\Psi) \tag{2.2}$$

and τ defines the $\mathbb{Z}^d$-action on $\mathcal{T}$ by $(\tau_{\mathbf{i}}\Psi)_{\mathbf{j}} = \Psi_{\mathbf{i}+\mathbf{j}}$. The pair $(\mathcal{A}, \mathcal{T})$ defines the coupled map lattice dynamical system.

To proceed we need to make assumptions on f. We suppose the coupling is weak and local, i.e. that $\mathcal{F}_i$ depends weakly on $\Psi_{\mathbf{j}}$ for $\mathbf{j}$ far away from $\mathbf{i}$. A convenient way to encode this is to assume f is holomorphic with derivatives with respect to $\Psi_{\mathbf{i}}$ decaying rapidly with $\mathbf{i}$. This condition can be formalized as follows. Given two positive constants α and β, let $\mathbb{T}_{\mathbf{i},\alpha,\beta} \subset \mathbb{C}^2/\mathbb{Z}^2$ be the complex neighborhood of $\mathbb{T}_{\mathbf{i}} = \mathbb{R}^2/\mathbb{Z}^2 \subset \mathbb{C}^2/\mathbb{Z}^2$ defined by $|\mathrm{Im}\,\Psi_{\mathbf{i},j}| < \alpha e^{|\mathbf{i}|\beta}$, $j = 1, 2$, where $\Psi_{\mathbf{i}} = (\Psi_{\mathbf{i},1}, \Psi_{\mathbf{i},2}) \in \mathbb{C}^2/\mathbb{Z}^2$ and $|\mathbf{i}| = \sum_{k=1}^{d} |\mathbf{i}_k|$. Moreover, let $\mathcal{R}$ be the Cartesian product over $\mathbf{i}$ of the $\mathbb{T}_{\mathbf{i},\alpha,\beta}$. If $\mathcal{O}$ is the space of holomorphic functions $f : \mathcal{R} \to \mathbb{C}$, equipped with the norm

$$\|f\|_\infty = \sup_{\Psi\in\mathcal{R}} |f(\Psi)| \tag{2.3}$$

we will consider the dynamical system, defined in (2.1) and (2.2), for $f \in \mathcal{O}$ with $\|f\|_\infty$ sufficiently small.

This infinite-dimensional dynamical system will be studied via finite-dimensional approximations which we now define. Letting $\mathcal{R}_N$ be the Cartesian product of the $\mathbb{T}_{\mathbf{i}\alpha,\beta}$ for $\mathbf{i} \in \Omega_N$ and given an $f \in \mathcal{O}$ we let f_N be the map defined on $\mathcal{R}_N$ given by

$$f_N(\Psi) = f(\Psi^p)$$

where $\Psi^p \in \mathcal{R}$ is obtained by extending $\Psi \in \mathcal{R}_N$ periodically to $\mathcal{R}$. We define the finite-dimensional approximation $\mathcal{A}_N$ to $\mathcal{A}$ by (2.1) and (2.2) where τ is the action of translations modulo $((2N-1)\mathbb{Z})^d$, i.e. we impose periodic boundary conditions on Ω_N. Observe that $\mathcal{A}$ maps the set $\mathcal{P}_N \subset \mathcal{R}$ of periodic points of period N to itself. Thus, identifying $\mathcal{R}_N$ with $\mathcal{P}_N$ we have

$$\mathcal{A}_N \equiv \mathcal{A}|_{\mathcal{P}_N}. \tag{2.4}$$

We define the SRB measure for $\mathcal{A}_N$ (respectively $\mathcal{A}$) to be the weak limit of $(1/n)\sum_{k=1}^{n} \mathcal{A}_N^k m_N$ $\big((1/n)\sum_{k=1}^{n} \mathcal{A}^k m\big)$ as $n \to \infty$ of the normalized Lebesgue measure m_N (m) on $\mathcal{T}_N$ $(\mathcal{T})$ if such a limit exists. Our first result concerns the existence of an SRB measure for $\mathcal{A}$.

THEOREM 1. *There exists an $\varepsilon > 0$ independent from N such that given $f \in \mathcal{O}$ with $\|f\|_\infty \le \varepsilon$ the dynamical system $\mathcal{A}$ ($\mathcal{A}_N$ respectively) admits an SRB measure μ (μ_N). The weak limit of μ_N as $N \to \infty$ exists and is equal to μ. The measures μ_N and μ are C^∞-smooth in f in the ball $\|f\|_\infty < \varepsilon$ of $\mathcal{O}$ in the sense that $\int T\,d\mu$ is C^∞-smooth for any C^∞-smooth T depending on finitely many variables $\Psi_{\mathbf{i}}$.*

Remark. The existence of the $N \to \infty$ limit of the SRB measures has been proven before [**JP**], with less stringent regularity assumptions than here. However, we need more detailed structure of the measures and have to go through the construction.

Let $\mathbf{P}$ be the projection of $\mathcal{T}_N$ to the torus at origin $\mathbb{T}$ and $\mathbf{P}^*\mu_N$ the induced projection of μ_N on $\mathbb{T}$. We want to address the question of whether this projection is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$.

Definition. $\mathcal{A}_N$ is degenerate if for all $\Psi \in \mathcal{T}_N$ the unstable manifold of Ψ is a Cartesian product of curves $\gamma_{\mathbf{i}}(\Psi)$ lying on the $\mathbf{i}$th torus.

An example of a degenerate map is the uncoupled map: in this case the curve $\gamma_{\mathbf{i}}(\Psi, \xi) = \Psi_{\mathbf{i}} + e^+\xi$ for $\xi \in \mathbb{R}$ where $Ae^+ = \Lambda_+ e^+$ with $\Lambda_+ > 1$. More generally if we choose $f(\Psi) = g(\Psi)e^+$ with $g : \mathcal{T} \to \mathbb{R}$ it is easy to see that the map $\mathcal{A}$ given by (2.1) and (2.2) with such an f has the same unstable foliation as A. In this case we will say that $\mathcal{A}$ is *coupled through the unstable manifold*. We can characterize all degenerate coupled maps through the following proposition.

PROPOSITION 1. *$\mathcal{A}_N$ is degenerate if and only if there exists $x : \mathbb{T} \to \mathbb{T}$ such that $X \circ \mathcal{A}_N \circ X^{-1} = \tilde{\mathcal{A}}_N$ where $(X(\Psi))_{\mathbf{i}} = x(\Psi_{\mathbf{i}})$ and $\tilde{\mathcal{A}}_N$ is coupled through the unstable manifold.*

Our main result can be formulated as follows.

THEOREM 2. *For each $2 \le N \le \infty$ if $\mathcal{A}_N$ is not degenerate then the projected measures $\mathbf{P}^*\mu_N$ are absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$. Moreover, if $\mathcal{A}$ is degenerate, then $\mathcal{A}_N$ is degenerate for every N and if $\mathcal{A}$ is non-degenerate then $\mathcal{A}_N$ is non-degenerate for N large enough.*

We close this section with a remark concerning the fractality of μ_N. The Hausdorff dimension of μ_N will generically satisfy $\dim_{HD}\mu_N < \dim \mathcal{T}_N$. In fact from the Kaplan–Yorke formula [**FKYY**] one obtains the upper bound

$$\dim_{HD} \mu_N \le \dim \mathcal{T}_N + \frac{\mu_N(\eta)}{\lambda_{\min}} \tag{2.5}$$

where $\lambda_{\min}$ is the minimum Lyapunov exponent of $\mathcal{A}_N$ and $\eta(\Psi) = -\log(\det D\mathcal{A}_N(\Psi))$. Generically we expect that $\mu_N(\eta)/\lambda_{\min} \ge \delta \dim \mathcal{T}_N$ for some constant δ. Indeed it is easy to show that for a generic perturbation of A acting on $\mathbb{T}$ $\mu_1(\eta) > 0$, see [**BGM**]. Adding a small enough coupling we will have $\mu_N(\eta) \simeq N\mu_1(\eta)$ while $\lambda_{\min}$ is almost independent from N. Theorem 2 asserts then that, notwithstanding this extensive loss of dimensionality of the attractor, the projected SRB measure is still absolutely continuous.

3. *The conjugacy*

We start by constructing a conjugacy $X : \mathcal{T} \to \mathcal{T}$ of the coupled map $\mathcal{A}$ to the uncoupled one A:

$$X \circ A = \mathcal{A} \circ X \tag{3.1}$$

Observe that from (2.4) it follows that $X_N \equiv X|_{\mathcal{P}_N}$ conjugates $\mathcal{A}_N$ to A_N.

Given a map $x : \mathcal{T} \to \mathbb{R}^2$ let $\tau x : \mathcal{T} \to (\mathbb{R}^2)^{\mathbb{Z}^d}$ be defined by translations as $(\tau x)_{\mathbf{i}} = x \circ \tau_{-\mathbf{i}}$. With this notation we have that $\mathcal{F} = \tau f$. Hence, guided by translation invariance of our map $\mathcal{A}$ we look for a solution of (3.1) in the form $X = \mathrm{Id} + \tau x$ with x a solution of the equation

$$\mathbf{T}x = f(\mathrm{Id} + \tau x) \tag{3.2}$$

where $\mathbf{T}$ is the linear operator defined by

$$\mathbf{T}x = x \circ A - A \circ x. \tag{3.3}$$

From general theory, we expect that the solution x will not be a differentiable function of Ψ but only Hölder continuous. Given a function $g : \mathcal{T}_N \to \mathbb{R}^2$ let $\delta_{\mathbf{j}}$ denote the *Hölder derivative*

$$\delta_{\mathbf{j}} g(\Psi) = \sup_{v_{\mathbf{j}}} \frac{|g(\Psi + v_{\mathbf{j}}) - g(\Psi)|}{|v_{\mathbf{j}}|^{\gamma}}$$

where $\gamma < 1$ and the supremum runs over vectors having a non-zero component only at the $\mathbf{j}$th position and of length no larger than unity. From now on we fix $\gamma < 1$ and, to avoid cumbersome notation, do not indicate the dependence of the estimates in what follows on γ as well as on α and β. Moreover we will use C to indicate the constants that appear in all the estimates.

Let $\mathcal{E}$ be the Banach space of Hölder continuous maps $x : \mathcal{T} \to \mathbb{R}^2$ with norm

$$\|x\| = \|x\|_{\infty} + \sum_{\mathbf{j}} e^{(\beta/2)|\mathbf{j}|} \|\delta_{\mathbf{j}} x\|_{\infty} \tag{3.4}$$

where in this case $\|x\|_{\infty} = \sup_{\Psi \in \mathcal{T}} |x(\Psi)|$. We then have the following.

PROPOSITION 2. *There exists an $\varepsilon > 0$ such that given $f \in \mathcal{O}$ with $\|f\|_{\infty} \leq \varepsilon$, (3.2) has a unique solution in $\mathcal{E}$ with $\|x\| \leq C\|f\|_{\infty}$. Moreover, x is analytic in f in the ball $\|f\|_{\infty} < \varepsilon$.*

Proof. Let us define $\mathbf{H}x = \mathbf{T}^{-1} f(\mathrm{Id} + \tau x)$. We want to show that $\mathbf{H}$ is a contraction in the ball $B = \{x | \|x\| \leq R\|f\|_{\infty}\}$ for a suitable R.

It is easy to find an explicit representation for $\mathbf{T}^{-1}$. Let e^+, Λ^+ and e^-, Λ^- denote the two eigenvectors of the matrix A and the corresponding eigenvalues, with $\Lambda^+ > 1$ and $\Lambda^- = (\det A/\Lambda^+)$, where $|\det A| = 1$; e^+ and e^- are the unit vectors in the direction of the unstable and stable manifolds of A at each point $\psi \in \mathbb{T}^2$. Expressing vectors $v \in \mathbb{R}^2$ in this basis as $v = v_+ e^+ + v_- e^-$, we have

$$(\mathbf{T}^{-1}x)(\Psi) = \sum_{n=0}^{\infty} \Lambda_-^n x_+(A^{-n+1}\Psi) + \sum_{n=1}^{\infty} \Lambda_+^{-n} x_-(A^{n-1}\Psi). \tag{3.5}$$

From this expression it follows immediately that the norm of $\mathbf{T}^{-1}$ as an operator in $\mathcal{E}$ is bounded by

$$\|\mathbf{T}^{-1}\|_{L(\mathcal{E},\mathcal{E})} \leq \frac{4}{1 - \Lambda_+^{-(1-\gamma)}}. \tag{3.6}$$

We now claim that the function $h_x(\Psi) = f(\Psi + \tau x(\Psi))$ satisfies

$$\|h_x\| \leq C\|f\|_{\infty}, \quad \|h_x - h_y\| \leq C\|f\|_{\infty} \|x - y\|. \tag{3.7}$$

To prove the first inequality in (3.7) we write

$$|h_x(\Psi + v_{\mathbf{j}}) - h_x(\Psi)| = \sum_{\mathbf{k}} \int_0^1 dt\, \partial_{\mathbf{k}} f(\Psi^t)(v_{\mathbf{j},\mathbf{k}} + x(\tau_{-\mathbf{k}}(\Psi + v_{\mathbf{j}})) - x(\tau_{-\mathbf{k}}\Psi))$$

where $\Psi^t = \Psi + t v_{\mathbf{j}} + t\tau x(\Psi + v_{\mathbf{j}}) + (1-t)\tau x(\Psi)$ and $v_{\mathbf{j},\mathbf{k}}$ is the $\mathbf{k}$ component of $v_{\mathbf{j}}$. Then, using $|\partial_{\mathbf{k}} f| \le e^{-\beta|\mathbf{k}|}\|f\|_\infty$, which follows from (2.3), and

$$|x(\tau_{-\mathbf{k}}(\Psi + v_{\mathbf{j}})) - x(\tau_{-\mathbf{k}}\Psi)| \le \eta^\gamma \|\delta_{\mathbf{j}-\mathbf{k}} x\|_\infty, \tag{3.8}$$

where we set $\eta = |v_{\mathbf{j}}|$, we get

$$\begin{aligned}&\sum_{\mathbf{j}} e^{(\beta/2)|\mathbf{j}|}\eta^{-\gamma}|h_x(\Psi + v_{\mathbf{j}}) - h_x(\Psi)|\\ &\qquad \le \|f\|\left(\sum_{\mathbf{j}} e^{-(\beta/2)|\mathbf{j}|} + \sum_{\mathbf{jk}} e^{(\beta/2)|\mathbf{j}|} e^{-\beta|\mathbf{k}|}\|\delta_{\mathbf{j}-\mathbf{k}} x\|_\infty\right).\end{aligned} \tag{3.9}$$

From (3.4) we infer $\|\delta_{\mathbf{j}-\mathbf{k}} x\|_\infty \le e^{-(\beta/2)|\mathbf{k}-\mathbf{j}|}\|x\|$. Hence by a use of the triangle inequality the right-hand side of (3.9) is bounded by

$$C\|f\|_\infty + \|f\|_\infty\|x\| \sum_{\mathbf{k}} e^{-(\beta/2)|\mathbf{k}|} \le C\|f\|_\infty(1 + \|x\|). \tag{3.10}$$

The second inequality of (3.7) can be proven as follows. Observe that

$$h_x(\Psi) - h_y(\Psi) = \int_0^1 dt\, \partial_{\mathbf{k}} f(\Psi + \tau x(\Psi) + (1-t)\tau y(\Psi))(x(\tau_{\mathbf{k}}\Psi) - y(\tau_{\mathbf{k}}\Psi)) \tag{3.11}$$

so that

$$\|h_x - h_y\| \le \sum_{\mathbf{k}} \|\partial_{\mathbf{k}} f(\mathrm{Id} + \tau x + (1-t)\tau y)\|\|x - y\|. \tag{3.12}$$

Combining (2.3) with a Cauchy estimate we infer that, for Ψ real,

$$|\partial_{\mathbf{k}}\partial_{\mathbf{i}} f(\Psi)| \le e^{-\beta(|\mathbf{k}|+|\mathbf{j}|)}\|f\|_\infty. \tag{3.13}$$

Proceeding as above this implies that

$$\|\partial_{\mathbf{k}} f(\mathrm{Id} + \tau x + (1-t)\tau y)\| \le C e^{-\beta|\mathbf{k}|}\|f\|_\infty \tag{3.14}$$

and (3.7) follows. Equations (3.6) and (3.7) establish the contractive property for suitable R. By the Banach fixed point theorem we have a unique solution of (3.2) which is analytic in f. □

4. *The invariant manifolds*

In this section we will construct the two invariant manifolds $W^\pm(\Psi)$ defined, for every point $\Psi \in \mathcal{T}$, by the property

$$W^\pm(\Psi) = \left\{\Psi' \,\Big|\, \lim_{n\to\infty} |\mathcal{A}^{\mp n}\Psi - \mathcal{A}^{\mp n}\Psi'| = 0\right\}, \tag{4.1}$$

where $|\Psi| = \sup_{\mathbf{i}} |\Psi_{\mathbf{i}}|$. We observe again that the stable and unstable manifolds of $\mathcal{A}_N$ are given by the periodic points in $W^\pm(\Psi)$. We will give below a unified construction of these sets for $N \leq \infty$, $N = \infty$ referring to $W^{\pm}(\Psi)$. For convenience the N-dependence of the various objects will be suppressed whenever possible.

We shall look for $W^{\pm}(\Psi)$ in terms of an embedding

$$\xi \in \mathbb{R}^{\Omega_N} \to S^{\pm}_{\Psi}(\xi) \in (\mathbb{R}^2)^{\Omega_N} \tag{4.2}$$

(Ω_∞ means $\Omega = \mathbb{Z}^d$) such that the action of $\mathcal{A}$ is given by

$$\mathcal{A}S^{\pm}_{\Psi}(\xi) = S^{\pm}_{\mathcal{A}\Psi}(\tilde{\mathcal{L}}^{\pm}(\Psi)\xi) \tag{4.3}$$

where $\tilde{\mathcal{L}}^{\pm}(\Psi)$ are linear operators on $\mathbb{R}^{\Omega_N}$. For $N = \infty$ we mean by the latter the vector space $\ell_\infty(\mathbb{Z}^d)$.

We want to use (4.3) to study the regularity properties of $S^{\pm}$ as a function of Ψ, ξ and $\mathcal{A}$. We expect on general grounds $S^{\pm}$ to be at most C^α in Ψ. Thus, since $\mathcal{A}$ occurs in (4.3) coupled to Ψ, low regularity can be expected for S also as a function of $\mathcal{A}$. However, it will be convenient to have maximal regularity in $\mathcal{A}$ and this can be achieved by looking for the solution to (4.3) in the form

$$S^{\pm}_{\Psi}(\xi) = \Psi + \mathcal{X}^{\pm}(X^{-1}(\Psi), \xi) \tag{4.4}$$

where X is the conjugation constructed in §3. Equation (4.3) implies the following equation for $\mathcal{X}^{\pm}$:

$$\mathcal{A}(X(\Psi) + \mathcal{X}^{\pm}(\Psi, \xi)) = X(A\Psi) + \mathcal{X}^{\pm}(A\Psi, \mathcal{L}^{\pm}(\Psi)\xi) \tag{4.5}$$

where $\mathcal{L} = \tilde{\mathcal{L}} \circ X$ and the previous problem is clearly not present. Indeed, we will show that (4.5) has a solution $\mathcal{X}^{\pm}$ that is *analytic* in f and in ξ as well.

To state the main result of this section we need to introduce the space where (4.5) will be solved. Let $\mathcal{D}_N$ be the complex domain $\mathcal{D}_N = \{\xi \mid |\xi_{\mathbf{i}}| < 1, \forall \mathbf{i} \in \Omega_N\}$. Let $\mathcal{B}$ be the Banach space of maps $\mathcal{X} : \mathcal{T}_N \times \mathcal{D}_N \to (\mathbb{C}^2)^{\Omega_N}$ which are Hölder continuous in Ψ and analytic in ξ equipped with the norm

$$\|\mathcal{X}\| = \sup_{\mathbf{i}} \left(\|\mathcal{X}_{\mathbf{i}}\|_\infty + \sum_{\mathbf{j}} e^{(\beta/4)|\mathbf{i}-\mathbf{j}|} \|D_{\mathbf{j}}\mathcal{X}_{\mathbf{i}}\|_\infty \right) \tag{4.6}$$

where $D_{\mathbf{j}} = (\delta_{\mathbf{j}}, \partial_{\xi_{\mathbf{j}}})$ and the infinity norm is intended in both Ψ and ξ for $\Psi \in \mathcal{T}_N$ and $\xi \in \mathcal{D}_N$. The following proposition describes the local stable and unstable manifolds.

PROPOSITION 3. *There exists an $\varepsilon > 0$, independent of $N \leq \infty$ such that given $f \in \mathcal{O}$ with $\|f\| \leq \varepsilon$ the local stable and unstable manifolds $W^{\pm}(\Psi)$ are given by real analytic embeddings*

$$S^{\pm}_{\Psi} : \mathcal{D}_N \to (\mathbb{R}^2)^{\Omega_N}.$$

$S^{\pm}_{\Psi}$ are translation-invariant: $S^{\pm}_{\tau_i\Psi}(\tau_i\xi) = S^{\pm}_{\Psi}(\xi)$ and are given by (4.4) with $\mathcal{X}^{\pm} \in \mathcal{B}$ and

$$\|\mathcal{X}^{\pm} - \Lambda_{\pm}\xi\| \leq C\|f\|_\infty.$$

Moreover $\mathcal{X}^{\pm}$ are analytic functions of f in the ball $\|f\|_\infty < \varepsilon$ of the Banach space $\mathcal{O}$.

To describe the global result let $\mathcal{D}$ be the Banach space of C^α maps $\mathcal{L}$ from $\mathcal{T}_N$ to the linear operators on $\mathbb{R}^{\Omega_N}$ equipped with the norm

$$\|\mathcal{L}\| = \sup_{\mathbf{i}} \left(\sum_{\mathbf{j}} e^{(\beta/4)|\mathbf{i}-\mathbf{j}|} \|\mathcal{L}_{\mathbf{ij}}\|_\infty + \sum_{\mathbf{jk}} e^{(\beta/4)|\mathbf{i}-\mathbf{k}|} \|\delta_{\mathbf{k}} \mathcal{L}_{\mathbf{ij}}\|_\infty \right). \tag{4.7}$$

PROPOSITION 4. *With the assumptions of Proposition 3, the global stable and unstable manifolds $W^\pm(\Psi)$ are given as real analytic embeddings*

$$S^\pm_\Psi : \mathbb{R}^{\Omega_N} \to (\mathbb{R}^2)^{\Omega_N}$$

that satisfy (4.5) with $\mathcal{L} \in \mathcal{D}$ and

$$\|\mathcal{L}^\pm - \Lambda_\pm\| \le C\|f\|_\infty. \tag{4.8}$$

Moreover, $S^\pm_\Psi$ can be extended to a complex neighborhood of $(\mathbb{R}^2)^{\Omega_N}$.

The rest of this section contains the proofs of these two propositions.

We start by separating the linear part in ξ from the rest in $\mathcal{X}^\pm(\Psi, \xi)$, i.e. we write

$$\mathcal{X}^\pm(\Psi, \xi) = \chi^\pm(\Psi)\xi + \bar{\mathcal{X}}^\pm(\Psi, \xi). \tag{4.9}$$

Observe that $\chi^\pm(\Psi)$ is a linear map from $\mathbb{R}^{\Omega_N}$ to $\mathrm{T}_\Psi \mathcal{T}_N$. We will choose as a basis on $\mathrm{T}_\Psi \mathcal{T}_N$ the one formed by the vectors $e^-_{\mathbf{i}}$ and $e^+_{\mathbf{i}}$.

The matrix $\chi^\pm(\Psi)$ satisfies the equation:

$$\mathcal{A}\chi^\pm(\Psi) - \chi^\pm(A\Psi)\mathcal{L}^\pm = D\mathcal{F}(X(\Psi))\chi^\pm(\Psi). \tag{4.10}$$

From now on we will consider explicitly only the unstable $(+)$ case and drop the $+$ superscript. Identical considerations hold for the stable manifold. It is easy to see that (4.10) alone cannot fix uniquely χ and $\mathcal{L}$. In fact if the pair $\chi(\Psi)$, $\mathcal{L}(\Psi)$ is a solution of (4.10) then, given any non-vanishing function $l : \mathcal{T}_N \to \mathbb{R}$,

$$\chi'(\Psi) = l(\Psi)\chi(\Psi), \quad \mathcal{L}'(\Psi) = \frac{l(\mathcal{A}\Psi)}{l(\Psi)} \mathcal{L}(\Psi) \tag{4.11}$$

is also a solution. To resolve the above ambiguity we fix $\chi_+ = \mathrm{Id}$ where the subscript $+$ refers to the component along the unstable directions and, with a slight abuse, we denote the $-$ component χ_- by χ. Thus χ is now an $\Omega_N \times \Omega_N$ matrix. Writing the matrix $H(\Psi) = D\mathcal{F}(X(\Psi))$ in the $\pm$ basis as

$$D\mathcal{A} = \begin{pmatrix} \Lambda_+ \mathrm{Id} + H_{++} & H_{+-} \\ H_{-+} & \Lambda_+^{-1}\mathrm{Id} + H_{--} \end{pmatrix} \tag{4.12}$$

it follows that

$$\Lambda_+ + H_{++} + H_{+-}\chi(\Psi) - \mathcal{L}(\Psi) = 0, \tag{4.13}$$

$$H_{-+} + (\Lambda_+^{-1} + H_{--})\chi(\Psi) - \chi(A\Psi)\mathcal{L}(\Psi) = 0. \tag{4.14}$$

Now setting

$$\mathcal{L}(\Psi) = \Lambda_+ \mathrm{Id} + \bar{\mathcal{L}}(\Psi) \tag{4.15}$$

we can solve (4.13) for $\bar{\mathcal{L}}(\Psi)$:

$$\bar{\mathcal{L}}(\Psi) = H_{++} + H_{+-}\chi(\Psi) \tag{4.16}$$

and substituting this in (4.14) we get

$$\mathbf{T}_1\chi(\Psi) = H_{--}\chi(\Psi) + H_{-+} - \chi(A\Psi)H_{++} - \chi(A\Psi)H_{++}\chi(\Psi) \equiv F(\chi, \Psi) \tag{4.17}$$

where $\mathbf{T}_1$ is the operator

$$(\mathbf{T}_1\chi)(\Psi) = \Lambda_+\chi(A\Psi) - \Lambda_+^{-1}\chi(\Psi). \tag{4.18}$$

We solve this equation in the Banach space $\mathcal{D}$ with the norm equation (4.7). The inverse of $\mathbf{T}_1$ is given by

$$\mathbf{T}_1^{-1}\chi(\Psi) = \sum_{n=0}^{\infty} \Lambda_+^{-2n-1}\chi(A^{-n-1}\Psi) \tag{4.19}$$

from which follows that $\mathbf{T}_1$ is a bounded operator in $\mathcal{D}$. Note that due to the extra power of Λ_+^{-1} compared to (3.5) we could work in C^1. This gain is not useful because $F(\chi, \Psi) \in C^\alpha$.

The solution of (4.17) proceeds analogously to what was done in the previous section. Writing it as $\chi = \mathbf{T}_1^{-1}F(\chi)$ we show the right-hand side is contraction in $\|\chi\| \le C\epsilon_0$. This follows in a straightforward fashion using the following lemmas.

LEMMA 1. *$\mathcal{D}$ is a Banach algebra:*

$$\|\chi\eta\| \le 2\|\chi\|\|\eta\|. \tag{4.20}$$

Proof. The claim follows from the simple estimates

$$\sum_{\mathbf{j}} e^{(\beta/2)|\mathbf{i}-\mathbf{j}|}|(\chi\eta)_{\mathbf{ij}}| \le \sum_{\mathbf{jl}} e^{(\beta/2)(|\mathbf{i}-\mathbf{l}|+|\mathbf{l}-\mathbf{j}|)}|\chi_{\mathbf{il}}||\eta_{\mathbf{lj}}| \le \|\chi\|\|\eta\| \tag{4.21}$$

and in a similar manner

$$\begin{aligned}\sum_{\mathbf{jk}} & e^{(\beta/2)|\mathbf{i}-\mathbf{k}|}|\partial_{\mathbf{k}}(\chi\eta)_{\mathbf{ij}}| \\ &\le \sum_{\mathbf{jkl}} (e^{(\beta/2)|\mathbf{i}-\mathbf{k}|}|\partial_{\mathbf{k}}\chi_{\mathbf{il}}||\eta_{\mathbf{lj}}| + e^{(\beta/2)|\mathbf{i}-\mathbf{l}|}|\chi_{\mathbf{il}}|e^{(\beta/2)|\mathbf{l}-\mathbf{k}|}|\partial_{\mathbf{k}}\eta_{\mathbf{lj}}|) \\ &\le 2\|\chi\|\|\eta\|.\end{aligned} \tag{4.22}$$

□

LEMMA 2. *For $i, j = \pm$ we have $H_{i,j} \in \mathcal{D}$.*

Proof. Note first that from (2.3) we get

$$|\partial_{\mathbf{k}}\mathcal{F}_{\mathbf{i}}(\Psi)| \le C\|f\|_\infty e^{-\beta|\mathbf{i}-\mathbf{k}|} \tag{4.23}$$

and

$$|\partial_{\mathbf{l}}\partial_{\mathbf{k}}\mathcal{F}_{\mathbf{i}}(\Psi)| \le C\|f\|_\infty e^{-\beta(|\mathbf{i}-\mathbf{k}|+|\mathbf{i}-\mathbf{l}|)} \tag{4.24}$$

for $\Psi \in \mathcal{R}$. It follows that

$$|\delta_{\mathbf{k}}\mathcal{H}_{\mathbf{i},\mathbf{j}}(\Psi)| = \left|\sum_{\mathbf{l}} \partial_{\mathbf{j}}\partial_{\mathbf{l}}\mathcal{F}_{\mathbf{i}}|_{X(\Psi)}\delta_{\mathbf{k}}X_{\mathbf{l}}(\Psi)\right| \le C\alpha^{-1}\|f\|_\infty^2 e^{-(\beta/2)|\mathbf{i}-\mathbf{k}|}. \tag{4.25}$$

These are summable when multiplied by the exponential factors in our norm. □

We can summarize the above discussion in the following proposition.

PROPOSITION 5. *There exists an ε such that given $f : \mathcal{R} \to \mathbb{R}^2$ with $\|f\|_\infty \le \varepsilon$, (4.10) has a unique solution $\chi = (1, \chi_-)$ with $\chi_- \in \mathcal{D}$ and $\|\chi_-\| \le C\|f\|_\infty$. $\mathcal{L}$ is given by (4.15) with $\|\bar{\mathcal{L}}\| \le C\|f\|_\infty$. Moreover χ and $\mathcal{L}$ are analytic in f in the ball $\|f\|_\infty < \varepsilon$.*

Let us finally consider the remainder $\bar{\mathcal{X}}$ in (4.9). Using (4.10) we deduce

$$\bar{\mathcal{X}}(A\Psi, \mathcal{L}(\Psi)\xi) - A\bar{\mathcal{X}}(\Psi, \xi) = G(\Psi, \xi, \bar{\mathcal{X}}) \tag{4.26}$$

where

$$G(\Psi, \xi, \bar{\mathcal{X}}) \equiv \mathcal{F}(X(\Psi) + \chi(\Psi)\xi + \bar{\mathcal{X}}(\Psi, \xi)) - \mathcal{F}(X(\Psi)) - D\mathcal{F}(X(\Psi))\chi(\Psi)\xi. \tag{4.27}$$

Let $\mathbf{T}_2$ be the operator

$$\mathbf{T}_2\bar{\mathcal{X}}(\Psi, \xi) = \bar{\mathcal{X}}(A\Psi, \mathcal{L}(\Psi)\xi) - A\bar{\mathcal{X}}(\Psi, \xi). \tag{4.28}$$

Thus we need to solve the equation

$$\bar{\mathcal{X}} = \mathbf{T}_2^{-1}G(\bar{\mathcal{X}}) \tag{4.29}$$

in the Banach space $\mathcal{B}$ with norm given by (4.6). First we need to control the inverse of $\mathbf{T}_2$ given formally by

$$(\mathbf{T}_2^{-1}\bar{\mathcal{X}})(\Psi, \xi) = \sum_{n=0}^{\infty} A^n \bar{\mathcal{X}}(A^{-n-1}\Psi, \widehat{\mathcal{L}}^n(\Psi)\xi) \tag{4.30}$$

where $\widehat{\mathcal{L}}^n(\Psi) = \prod_{i=1}^{n-1} \mathcal{L}(A^{-i}\Psi)$. Recall that $\bar{\mathcal{X}}$ vanishes at $\xi = 0$ together with its first derivatives, i.e. we want to solve our equation in the closed subspace $\mathcal{B}_0$ of $\mathcal{B}$ of functions with this property.

LEMMA 3. *The map*

$$\mathbf{F} : \bar{\mathcal{X}} \to A\bar{\mathcal{X}}(A^{-1}\Psi, \mathcal{L}(\Psi)^{-1}\xi) \tag{4.31}$$

is a bounded map from $\mathcal{B}_0$ into itself with norm strictly less than one.

Proof. From $\mathcal{L} = \Lambda_+ + \bar{\mathcal{L}}$ and $\|\bar{\mathcal{L}}\| \le C\|f\|_\infty$ we infer that if $\xi \in \mathcal{D}_N$ then $\mathcal{L}(\Psi)^{-1}\xi \in \rho\mathcal{D}_N$ with

$$\rho = (\Lambda_+ - C\|f\|_\infty)^{-1}.$$

Hence by a Cauchy estimate, taking into account that $\bar{\mathcal{X}}(\Psi, \xi)$ vanish to second order for $\xi = 0$, we get

$$\|\mathbf{F}\bar{\mathcal{X}}_i\|_\infty \le \lambda\|\bar{\mathcal{X}}_i\|_\infty$$

for $\lambda = \Lambda_+\rho^2 < 1$ provided $\|f\|_\infty$ is chosen small enough.

For the second factor occurring in the norm equation (4.6) we write

$$\begin{aligned}
&\sum_{\mathbf{j}} e^{(\beta/4)|\mathbf{i}-\mathbf{j}|}\|D_{\mathbf{j}}\bar{\mathcal{X}}_{\mathbf{i}}(A^{-1}\Psi, \mathcal{L}(\Psi)^{-1}\xi)\|_\infty \\
&\quad \le \rho^2\Lambda_+^\gamma \sum_{\mathbf{j}} e^{(\beta/4)|\mathbf{i}-\mathbf{j}|}\|\delta_{\mathbf{j}}\bar{\mathcal{X}}_{\mathbf{i}}(\Psi, \xi)\|_\infty \\
&\qquad + \rho \sum_{\mathbf{k},\mathbf{l},\mathbf{j}} e^{(\beta/4)|\mathbf{k}-\mathbf{j}|}\|\delta_{\mathbf{j}}(\mathcal{L}(\Psi)^{-1})_{\mathbf{k},\mathbf{l}}\|_\infty e^{(\beta/4)|\mathbf{k}-\mathbf{i}|}\|\partial_{\xi_{\mathbf{k}}}\bar{\mathcal{X}}_{\mathbf{i}}(\Psi, \xi)\|_\infty \\
&\qquad + \rho \sum_{\mathbf{j},\mathbf{k}} e^{(\beta/4)|\mathbf{k}-\mathbf{j}|}\|(\mathcal{L}(\Psi)^{-1})_{\mathbf{k},\mathbf{j}}\|_\infty e^{(\beta/4)|\mathbf{i}-\mathbf{k}|}\|\partial_{\xi_{\mathbf{k}}}\bar{\mathcal{X}}_{\mathbf{i}}(\Psi, \xi)\|_\infty
\end{aligned}$$

where the factors ρ^2 and ρ come from a Cauchy estimate on $\rho\mathcal{D}_N$. By the definitions of the norms (4.6) and (4.7) the sums may be bounded by

$$(\rho^2\Lambda_+^\gamma + \rho\|\mathcal{L}(\Psi)^{-1}\|)\|\bar{\mathcal{X}}\|$$

and since

$$\|\mathcal{L}(\Psi)^{-1}\| \leq (\Lambda_+ - C\|f\|_\infty)^{-1}$$

the claim follows with $\|f\|$ small enough. □

Hence $\mathbf{T}_2$ has a bounded inverse in $\mathcal{B}_0$ as long as $\gamma < 1$.

Next we turn to the study of $\|G\|$. Note that G is well defined: the argument of $\mathcal{F}$ in (4.26) is in its analyticity domain if $C\|f\|_\infty < \alpha$. Moreover, we want to prove that

$$\|G(\bar{\mathcal{X}})\| \leq C\|f\|_\infty\|\bar{\mathcal{X}}\|, \quad \|G(\bar{\mathcal{X}}) - G(\bar{\mathcal{Y}})\| \leq C\|f\|_\infty\|\bar{\mathcal{X}} - \bar{\mathcal{Y}}\| \tag{4.32}$$

so that we can conclude our proof, invoking again the Banach fixed-point theorem.

To prove the above estimates we must bound both the derivatives in ξ and the Hölder derivative in Ψ of $\mathcal{G}$. It is easy to see that the ξ derivatives bound follows easily from Cauchy type estimates like (4.23) and (4.24). To bound the Hölder derivative in Ψ we observe that for both of the above estimates it is enough to study the first term in the definition (4.27) since good estimates were already proven on the other two terms while proving the existence of X and χ. To this end, using $\mathcal{H}(\Psi, \bar{\mathcal{X}}) = \mathcal{H}(X(\Psi) + \chi(\Psi)\xi + \bar{\mathcal{X}}(\Psi, \xi))$, we can write

$$\begin{aligned}
&\mathcal{H}_{\mathbf{i}}(\Psi, \bar{\mathcal{X}}) - \mathcal{H}_{\mathbf{i}}(\Psi + \delta v_{\mathbf{i}}, \bar{\mathcal{X}}) \\
&\quad = \sum_{\mathbf{k}} \int_0^1 dt\, \partial_{\mathbf{k}}\mathcal{H}_{\mathbf{i}}(\Psi^t) \cdot (v_{\mathbf{j},\mathbf{k}} + (X(\Psi + v_{\mathbf{j}}) - X(\Psi)) \\
&\qquad + (\chi(\Psi + v_{\mathbf{j}})\xi - \chi(\Psi)\xi) + (\bar{\mathcal{X}}(\Psi + v_{\mathbf{j}}, \xi) - \bar{\mathcal{X}}(\Psi, \xi))),
\end{aligned} \tag{4.33}$$

where we have set

$$\Psi^t = tv_{\mathbf{j},\mathbf{k}} + t(X(\Psi+v_{\mathbf{j}}) + \chi(\Psi+v_{\mathbf{j}})\xi + \bar{\mathcal{X}}(\Psi+v_{\mathbf{j}}, \xi)) + (1-t)(X(\Psi) + \chi(\Psi)\xi + \bar{\mathcal{X}}(\Psi, \xi)) \tag{4.34}$$

and proceed like in (3.10). The second inequality follows from

$$\begin{aligned}
&\mathcal{H}(\Psi, \xi, \bar{\mathcal{X}}) - \mathcal{H}(\Psi, \xi, \bar{\mathcal{Y}}) \\
&\quad = \int_0^1 dt\, \partial_{\mathbf{k}}\mathcal{F}(\Psi + X(\Psi) + \chi(\Psi)\xi + t\bar{\mathcal{X}}(\psi) + (1-t)\bar{\mathcal{Y}}(\Psi))(\bar{\mathcal{X}}(\Psi) - \bar{\mathcal{Y}}(\Psi))
\end{aligned} \tag{4.35}$$

and again we can conclude like in (3.12). By the Banach fixed-point theorem we get a solution of (4.29). Combining the solution of (4.10) and (4.29) we obtain a proof of Proposition 3.

To prove Proposition 4 note that the analyticity domain of $\mathcal{X}^+$ in ξ is independent of Ψ. Equation (4.5) implies

$$\mathcal{X}(\Psi, \xi) = \mathcal{A}(X(A^{-1}\Psi) + \mathcal{X}(A^{-1}\Psi, \mathcal{L}(A^{-1}\Psi)^{-1}\xi)) - X(\Psi) \tag{4.36}$$

so that by Lemma 3 the right-hand side provides analytic continuation of the left-hand side to $\rho\mathcal{D}_N$ with $\rho = (\Lambda_+ - C\|f\|_\infty)$. Iterating this formula n times we expand the domain of $\mathcal{X}^+$ as long as $X(A^{-1}\Psi) + \mathcal{X}^\pm(A^{-1}\Psi, \mathcal{L}^+(A^{-1}\Psi)^{-1}\xi)$ is in the analyticity domain of $\mathcal{A}$. Since $\mathcal{A} = A + \mathcal{F}$ the imaginary part of $\mathcal{X}$ may expand each step by a factor $\Lambda_+ + C\varepsilon_0$. Hence, for $\mathrm{Re}\,\xi \in \rho_n\mathcal{D}_N$ with $\rho_n = (\Lambda_+ - C_1\varepsilon_0)^n$, we can take $\mathrm{Im}\,\xi \in r_n\mathcal{D}_N$ with $r_n = (\Lambda_+ + C_2\varepsilon_0)^{-n}$. Thus $\mathcal{X}^+$ is analytic in ξ in such a neighborhood of $\mathbb{R}^{\Omega_N}$. Furthermore, since $W^\pm_{\mathcal{F}}(\Psi) = X_{\mathcal{F}}(W^\pm_0(\Psi))$, as follows immediately from the definition of the unstable manifold, the continuity of X and density of $W^+_0(\Psi)$ imply that $W^+_{\mathcal{H}}(\Psi)$ is dense in $\mathcal{T}_N$.

5. *The SRB measure*

The SRB measure is constructed in a standard way using a Markov partition. Since we want to have a construction uniform in N and also keep track of analyticity properties in that limit we cannot refer directly to standard constructions. However, we assume the reader is familiar with the various standard definitions concerning Markov partitions and thermodynamic formalism and will use them freely without comment [**R78**].

Let $Q = \{Q_i\}_{1,\dots,m}$ be a Markov partition of the two-torus $\mathbb{T}$ corresponding to the linear map A. We recall that the Q_i are standard rectangles in $\mathbb{R}^2$ with sides parallel to the vectors $e^\pm$.

Let $S_N = \{1, \dots, m\}^{\Omega_N}$. Then $\mathbf{Q} = \{\mathbf{Q}_s\}_{s\in S_N}$ where $\mathbf{Q}_s = \times_{\mathbf{i}\in\Omega_N} Q_{s(\mathbf{i})}$ is a Markov partition for A acting on $\mathcal{T}_N$ and

$$\mathcal{Q} = \{\mathcal{Q}_s\}_{s\in S_N}, \quad \mathcal{Q}_s = X(\mathbf{Q}_s) \tag{5.1}$$

is a Markov partition for $\mathcal{A}$.

As usual, a Markov partition allows us to conjugate $\mathcal{A}$ to a subshift of finite type on a symbol sequence space. Let $\bar{\Sigma}_N = S_N^{\mathbb{Z}}$ and denote its elements by $\sigma = \{\sigma_i\}_{i\in\mathbb{Z}}$ where $\sigma_i \in S_N$ is written as $\sigma_i = (\sigma_i(\mathbf{j}))_{\mathbf{j}\in\Omega_N}$. The fact that $\mathcal{Q}$ is a Markov partition implies that the set

$$\mathcal{P}(\sigma) = \bigcap_{i\in\mathbb{Z}} \mathcal{A}^{-i}(\mathcal{Q}_{\sigma_i}) \tag{5.2}$$

contains at most one point. Let Σ_N be the set of all σ such that $\mathcal{P}(\sigma)$ contains exactly one point (we will call this point $\mathcal{P}(\sigma)$ with a small abuse of notation). The Markov property of $\mathcal{Q}$ and the way we constructed it imply that there exists an $m \times m$ matrix M with $M_{ij} \in \{0, 1\}$ such that $\sigma \in \Sigma_N$ if and only if $M_{\sigma_i(\mathbf{j}),\sigma_{i+1}(\mathbf{j})} = 1$ for every $i \in \mathbb{Z}$ and $\mathbf{j} \in \Omega_N$. We equip Σ_N with the metric

$$d(\sigma, \sigma') = \sum_{i,\mathbf{j}} 2^{-(|\mathbf{i}|+|\mathbf{j}|)}|\sigma_i(\mathbf{j}) - \sigma'_i(\mathbf{j})|. \tag{5.3}$$

PROPOSITION 6. *The map $\mathcal{P} : \Sigma_N \to \mathcal{T}_N$ is given by $\mathcal{P}_\mathbf{i} = p \circ \tau_{-\mathbf{i}}$ where $p : \Sigma_N \to \mathbb{T}$ and*

$$|p(\sigma) - p(\sigma')| \le Cd(\sigma, \sigma')^\eta$$

for a suitable Hölder exponent η. Moreover, $\mathcal{P}$ conjugates $\mathcal{A}$ to the shift $\tilde{\tau}$ on Σ_N, where $(\tilde{\tau}\sigma)_i = \sigma_{i-1}$.

Proof. Let $\mathcal{P}_0(\sigma)$ be the map associated with A. It is clear that $\mathcal{P}_0(\sigma) = p_0 \circ \tau_{-\mathbf{i}}$ and that p_0 depends only on the value of σ at the origin $\mathbf{o}$ of $\mathbb{Z}^d$. For this map the time part of the estimate is a simple consequence of the hyperbolicity of A. Our theorem follows immediately from the fact that $\mathcal{P}(\sigma) = X(\mathcal{P}_0(\sigma))$ and the Hölder continuity of X proved in §3. □

Observe that if we consider the metric on $\mathcal{T}$ given by

$$d(\Psi, \Psi') = \sum_{\mathbf{j}} 2^{-|\mathbf{j}|} |\Psi_{\mathbf{j}} - \Psi'_{\mathbf{j}}|, \tag{5.4}$$

then $\mathcal{P}$ is a Hölder function from Σ to $\mathcal{T}$.

The SRB measure is constructed in the standard fashion by studying the Jacobian of the map $\mathcal{A}$ restricted to the unstable foliation. Recall that the local unstable manifold at Ψ is given by the embedding (4.2). We will use as a basis of the tangent space $TW^+(\Psi)$ the vectors $\partial_{\xi_{\mathbf{j}}}$, $\mathbf{j} \in \Omega_N$. In this basis the Jacobian of $\mathcal{A}$ restricted to the unstable foliation is given at the point Ψ by $\det \tilde{\mathcal{L}}(\Psi)$. Thus, let us define

$$\lambda^+(\Psi) = -\log\, \det(\Lambda_+^{-1} \mathcal{L}(X^{-1}(\Psi))) \tag{5.5}$$

where the constant Λ_+^{-1} was inserted for later convenience, and let

$$h^+(\sigma) = \lambda^+(\mathcal{P}(\sigma)).$$

PROPOSITION 7. *λ^+ and h^+ can be written as a sum of local functions as follows:*

$$\lambda^+(\Psi) = \sum_{\mathbf{i} \in \Omega_N} \lambda(\tau_{\mathbf{i}} \Psi) \tag{5.6}$$

and

$$h^+(\sigma) = \sum_{\mathbf{i} \in \Omega_N} h(\tau_{\mathbf{i}} \sigma) \tag{5.7}$$

with λ and h Hölder continuous with constants uniform in N. Furthermore,

$$|\lambda(\Psi) - \lambda(\Psi')| \le C \|f\|_\infty d(\Psi, \Psi')^\eta \tag{5.8}$$

and

$$|h(\sigma) - h(\sigma')| \le C \|f\|_\infty d(\sigma, \sigma')^\eta. \tag{5.9}$$

Proof. Writing

$$\lambda^+(\Psi) = \operatorname{Tr} \log(1 + \bar{\mathcal{L}}(X(\Psi))\Lambda_+^{-1}) = \operatorname{Tr} \sum_{i=1}^{\infty} \frac{(-1)^i}{i} \frac{\bar{\mathcal{L}}(X(\Psi))^i}{\Lambda_+^i}$$

we can define

$$\lambda(\Psi) = \sum_{i=1}^{\infty} \frac{(-1)^i}{i} \frac{(\bar{\mathcal{L}}(X(\Psi))^i)_{\mathbf{o},\mathbf{o}}}{\Lambda_+^N}.$$

From Lemma 1 and Proposition 3 we get $\|\lambda(\Psi)\|_\infty < C\|f\|_\infty$ and $\|\delta_{\mathbf{i}} \lambda(\Psi)\|_\infty < Ce^{-(\beta/4)|\mathbf{i}|} \|f\|_\infty$ from which (5.8) follows immediately. Equation (5.9) is an immediate consequence of (5.8) and Proposition 5. □

The SRB measure of our system will be given in terms of a Gibbs state on Σ_N. Let $\bar{m}$ be the maximum entropy measure on Σ_N and let us define the 'Hamiltonian'

$$H_T(\sigma) = \sum_{i=-T}^{T} h^+(\tilde{\tau}^i(\sigma)). \tag{5.10}$$

Moreover we set

$$\tilde{\mu}^T(d\sigma) = \frac{1}{Z_T} e^{H_T(\sigma)} \bar{m}(d\sigma) \tag{5.11}$$

where $Z_T = \int e^{H_T} \, d\bar{m}$.

PROPOSITION 8. *The weak limits*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{A}_N^k m_N = \mu_N, \quad \lim_{T\to\infty} \tilde{\mu}^T = \tilde{\mu}_N$$

exist and $\mu_N = \mathcal{P}\tilde{\mu}_N$. *Furthermore,* μ_N *and* $\tilde{\mu}_N$ *converge weakly to measures* μ *and* $\tilde{\mu}$ *as* $N \to \infty$ *and* $\mu = \mathcal{P}\tilde{\mu}$.

Proof. For any finite N the maps λ^+ and h^+ are Hölder continuous. For instance

$$|\lambda^+(\Psi) - \lambda^+(\Psi')| \leq \sum_{\mathbf{j}} |\lambda(\tau_{\mathbf{j}}\Psi) - \lambda(\tau_{\mathbf{j}}\Psi')| \leq C(N) \sup_{\mathbf{j}} d(\tau_{\mathbf{j}}\Psi, \tau_{\mathbf{j}}\Psi')^\eta$$

and the last distance is bounded by $C(N)d(\Psi, \Psi')$, as is readily seen from (5.4). The Bowen–Ruelle theorem [**Bo**] yields the claim for $(1/n)\sum_{k=1}^{n} \mathcal{A}_N^k m_N$.

The claim for $\tilde{\mu}^T$ can be proven similarly, but let us prove a more general result that comprises both the T and the N limits. Consider the Hamiltonian

$$H_{T,N}(\sigma) = \sum_{i=-T}^{T} \sum_{\mathbf{j}\in\Omega_N} h(\tau_{\mathbf{j}}\tilde{\tau}^i(\sigma)). \tag{5.12}$$

Given a $\sigma \in \Sigma_N$ let $\sigma^n \in \Sigma_N$ be defined as $\sigma_i^n(\mathbf{j}) = \sigma_i(\mathbf{j})$ for $|\mathbf{j}| \leq n$ and $\sigma_i^n(\mathbf{j}) = \sigma_i(0)$ for $|\mathbf{j}| > n$. Write

$$h(\sigma) = h(\sigma^0) + \sum_{n=1}^{n(N)} (h(\sigma^n) - h(\sigma^{n-1})) \equiv \sum_{n=0}^{n(N)} h_n(\sigma) \tag{5.13}$$

and then do a similar telescoping sum in the time direction† for each $h^n(\sigma)$ arriving at

$$h(\sigma) = \sum_R h_R(\sigma) \tag{5.14}$$

where R are sets of the form $\{(i, \mathbf{j}) \mid |i| \leq m, |\mathbf{j}| \leq n\} \in \mathbb{Z} \times \Omega_N$ and h_R depends on σ only through its restriction to R. The Hölder continuity expressed by (5.8) of h implies

$$|h_R| \leq C\|f\|_\infty e^{-cd(R)} \tag{5.15}$$

† Some care should be paid here to take into account the compatibility matrix M. This is a standard construction, see e.g. [**G99**].

where $d(R)$ is the diameter of R. For the full Hamiltonian we get now

$$H_{T,N}(\sigma) = \sum_R h_R(\sigma) \tag{5.16}$$

where R are rectangles similar to the ones appearing in (5.14) but centered arbitrarily in $[-T, T] \times \Omega_N$. For the existence of the limit

$$\lim_{N\to\infty} \lim_{T\to\infty} e^{H_{T,N}} \bar{m} \tag{5.17}$$

(in any order, indeed) we refer the reader to e.g. [**BK**] where it is proven in our set-up provided that $\|f\|_\infty$ is small enough. We should warn the reader that standard high-temperature expansion methods cannot be used when the interactions have a decay as in (5.15) where only the diameter of the set R occurs (one needs the volume of R). See [**BK**] for a discussion of these subtleties.

Finally we have to prove that $\lim_{N\to\infty} \mu_N = \mu$. To do this one can use the symbolic map $\mathcal{P}$. Some care should be paid to the fact that $\mathcal{P}$ is not one-to-one. Indeed the points on the set

$$\partial_\infty \mathcal{Q} = \bigcup_{n=-\infty}^{\infty} \bigcup_s \partial \mathcal{Q}_s$$

have more than one symbolic representation. Hence we need to show that for every s and N we have $\mu_N(\partial \mathcal{Q}_s) = 0$. For $N < \infty$ this is evident while for $N = \infty$ it follows from a standard argument [**JP**].

6. *Decomposition of the SRB measure*

6.1. *Coordinates on rectangles.* In order to study the projection of the SRB measure on finitely many tori we need to express it in terms of our parameterization of the stable and unstable manifolds constructed in §4. To do this we will introduce new coordinates on the rectangles $\mathcal{Q}_s$. For $\Psi \in \mathcal{Q}_s$ let

$$W_s^\pm(\Psi) = W^\pm(\Psi) \cap \mathcal{Q}_s. \tag{6.1}$$

Let us fix an arbitrary point ψ_i on each basic rectangle Q_i of the 2-torus. Observe that $Q_i = U_i \times S_i$ where U_i and S_i are segments in the direction of e^+ and e^-, respectively, containing ψ_i. We set $\bar{\Psi}_s = (\psi_{s(\mathbf{j})})_{\mathbf{j}\in\Omega} \in \mathbf{Q}_s$ and call $\Psi_s = X(\bar{\Psi}_s)$ the *center* of $\mathcal{Q}_s$. From the fact that $\mathcal{Q}_s$ is a rectangle we know that for every $\Psi \in \mathcal{Q}_s$ there is one and only one $\Psi' \in W_s^-(\Psi_s)$ such that $\Psi \in W_s^+(\Psi')$. Hence there exists a unique $\xi^- \in \mathbb{R}^{\Omega_N}$ such that $\Psi' = S^-_{\Psi_s}(\xi^-)$ and a unique $\xi^+ \in \mathbb{R}^{\Omega_N}$ such that $\Psi = S^+_{\Psi'}(\xi^+)$. Thus we have a one-to-one map $\Psi \in \mathcal{Q}_s \to (\xi^-, \xi^+) \in \mathbb{R}^{\Omega_N} \times \mathbb{R}^{\Omega_N}$ whose inverse we will, with slight abuse, denote by $\Psi^N(\xi^-, s, \xi^+)$, i.e.

$$\Psi^N(\xi^-, s, \xi^+) = S^+_{S^-_{\Psi_s}(\xi^-)}(\xi^+). \tag{6.2}$$

Ψ^N can be viewed as a continuous map $\mathcal{M}_N \to \mathcal{T}_N$ where $\mathcal{M}_N$ is a compact subset of $S_N \times \mathbb{R}^{\Omega_N} \times \mathbb{R}^{\Omega_N}$ given by

$$\mathcal{M}_N = \{(\xi^-, s, \xi^+) \mid s \in S_N, \xi^- \in I_N(s), \xi^+ \in J_N(s, \xi^-)\} \tag{6.3}$$

where

$$I_N(s) = (S^-_{\Psi_s})^{-1} W^-_s(\Psi_s) \tag{6.4}$$

and

$$J_N(s,\xi^-) = (S^+_{\Psi'})^{-1} W^+_s(\Psi'), \quad \Psi' = S^-_{\Psi_s}(\xi^-). \tag{6.5}$$

Denoting the points in $\mathcal{M}_N$ by m, we have by translation invariance (see Proposition 3)

$$\Psi^N_{\mathbf{i}}(m) = \Psi^N_0(\tau_{-\mathbf{i}}m).$$

It is easy to see from the properties of the maps $S^{\pm}_{\Psi}$ that there exists an r independent of N such that $\mathcal{M}_N \subset S_N \times C^N_r \times C^N_r = \widehat{\mathcal{M}}_N$ where C^N_r is the cube of side r centered at the origin of $\mathbb{R}^{\Omega_N}$.

Setting C^∞_r equal to the r-cube in $\mathbb{R}^\Omega$ and giving it the topology defined by the metric

$$d(\xi,\xi') = \sum_{\mathbf{j}} 2^{-|\mathbf{j}|}|\xi_{\mathbf{j}} - \xi'_{\mathbf{j}}| \tag{6.6}$$

and $S_\infty = \{1,\dots,m\}^\Omega$ with the metric

$$d(s,s') = \sum_{\mathbf{j}} 2^{-|\mathbf{j}|}|s(\mathbf{j}) - s'(\mathbf{j})| \tag{6.7}$$

we have that C^∞_r and S_∞ are compact metric spaces. We can view $\mathcal{M}$ as a compact subset of $\widehat{\mathcal{M}}$.

The following proposition summarizes the important properties of the function Ψ^N.

PROPOSITION 9. *There exists an r such that Ψ^N can be extended to a function from $\widehat{\mathcal{M}}_N$ to $\mathcal{T}$, still denoted by Ψ^N. For every $s \in S_N$, $\Psi^N(\xi^-,s,\xi^+)$ is one-to-one from $C^N_r \times C^N_r$ into its image. Moreover Ψ^N_0 converge as $N \to \infty$ uniformly to a Hölder continuous function Ψ_0. Finally, for each (ξ^-,s), $\Psi_0(\xi^-,s,\xi^+)$ is analytic in ξ^+ for $|\mathrm{Im}\,\xi_{\mathbf{i}}| < 1$.*

Proof. The extensions follows from the fact that ξ^+ and ξ^- are global coordinates on the unstable and stable manifolds. Moreover the image of $C^N_r \times C^N_r$ under Ψ^N is close to $\mathbf{Q}_s$ for every s, from which the one-to-one property follows.

The regularity property in $\xi^{\pm}$ immediately follows from Proposition 1 while the regularity in s is a consequence of the construction of the center Ψ_s, see definition after (6.1). □

Let us spell out the correspondence between the coordinates (ξ^-,s,ξ^+) and the symbolic representation. Define

$$C_s = \{\sigma | \sigma_0 = s\}, \tag{6.8}$$

i.e. the set of all the sequences σ that agree with s at the position 0. On C_s we have coordinates $\sigma^{\pm} \in S_N^{\mathbb{Z}^{\pm}} \equiv \Sigma^{\pm}_N$ where $\mathbb{Z}^{\pm}$ are the strictly positive (negative) integers and

$$(\sigma^-,\sigma^+) \to \sigma^- \vee s \vee \sigma^+$$

is one-to-one $\Sigma^-_N \times \Sigma^+_N \to C_s$. Clearly $\mathcal{P}(C_s) = \mathcal{Q}_s$ and given a point $\bar\sigma \in C_s$ with $\mathcal{P}(\bar\sigma) = \Psi$ then

$$\begin{aligned} W^+_s(\Psi) &= \{\mathcal{P}(\bar\sigma^-,s,\sigma^+) | \sigma^+ \in \Sigma^+_N\}, \\ W^-_s(\Psi) &= \{\mathcal{P}(\sigma^-,s,\bar\sigma^+) | \sigma^- \in \Sigma^-_N\}. \end{aligned} \tag{6.9}$$

Now the map Θ_N given by

$$\Theta_N(\sigma) = \Psi_N^{-1}(\mathcal{P}(\sigma))$$

gives the desired correspondence between the two coordinate systems.

6.2. *Decomposition in finite volume.* Our goal is to find a representation of the SRB measure in terms of the coordinates (ξ^-, s, ξ^+). Let us define

$$\nu_N = \Theta_N \tilde{\mu}_N.$$

We will decompose the measure $\tilde{\mu}_N$ in a convolution of different probability measures and then discuss their image under the map Θ_N. Since the volume N is kept fixed in this subsection, we will omit it in the notation. Recall that $\tilde{\mu} = \lim_{T\to\infty} \tilde{\mu}^T$ with

$$\tilde{\mu}^T(d\sigma) = \frac{1}{Z_T} e^{H_T(\sigma)} \bar{m}(d\sigma).$$

Write $\sigma = \sigma^- \vee s \vee \sigma^+$ and decompose the maximum entropy measure as

$$\bar{m}(d\sigma) = \bar{m}(d\sigma^-|s)\bar{m}(d\sigma^+|s)b(ds) \tag{6.10}$$

where $\bar{m}(d\sigma^\pm|s)$ is the measure $\bar{m}$ on $\Sigma_N^\pm$ conditioned on s and b is the counting measure on S_N. Similarly decompose the Hamiltonian

$$H_T(\sigma) = H_T^+(\sigma) + H_T^-(\sigma)$$

into terms depending mostly on the σ^+ or σ^-:

$$H_T^+(\sigma) = \sum_{i=1}^{T} h^+(\tau^i\sigma), \quad H_T^-(\sigma) = \sum_{i=0}^{T} h^+(\tau^{-i}\sigma). \tag{6.11}$$

Define on Σ_N^+ the probability measure, depending parametrically on s and σ^-:

$$\tilde{\mu}^T_{s,\sigma^-}(d\sigma^+) = \frac{1}{Z_T(s,\sigma^-)} e^{H^+(\sigma^-\vee s\vee\sigma^+)} \bar{m}(d\sigma^+|s) \tag{6.12}$$

where

$$Z^T(s,\sigma^-) = \int e^{H^+(\sigma^-\vee s\vee\sigma^+)} \bar{m}(d\sigma^+|s). \tag{6.13}$$

Let $\sigma_s = \sigma_s^- \vee s \vee \sigma_s^+$ be the symbolic representation of the center Ψ_s of $\mathcal{Q}_s$ and set

$$\tilde{\mathcal{I}}_T(\sigma) = H_T^-(\sigma) - H_T^-(\sigma^- \vee s \vee \sigma_s^+).$$

We can then write our measure as

$$\tilde{\mu}^T(d\sigma) = e^{\tilde{\mathcal{I}}_T(\sigma)} \tilde{\mu}^T_{s,\sigma^-}(d\sigma^+)\tilde{\mu}^T_s(d\sigma^-)b(ds) \tag{6.14}$$

where

$$\tilde{\mu}^T_s(d\sigma^-) = \frac{Z_T(s,\sigma^-)}{Z_T} e^{H_T^-(\sigma^-\vee s\vee\sigma_s^+)} \bar{m}(d\sigma^-|s). \tag{6.15}$$

The following proposition characterizes the images under Θ of μ^T_{s,σ^-}, μ^T_s and $\tilde{\mathcal{I}}_T$.

PROPOSITION 10.

(a) *The limit* $\tilde{\mu}_{s,\sigma_-} = \lim_{T\to\infty} \tilde{\mu}^T_{s,\sigma_-}$ *exists, and* $\nu_{s,\xi^-} = \Theta\mu_{s,\sigma^-}$ *is the normalized Lebesgue measure* $|J_N(s,\xi^-)|^{-1}\, d\xi^+$ *on* $J_N(s,\xi^-)$ *where* ξ^- *is given by* $\Theta(\sigma^- \vee s \vee \sigma_s^+) = (\xi^-, s, 0)$ *and* $|J_N(s,\xi^-)|$ *is the Lebesgue measure of* $J_N(s,\xi^-)$.

(b) *The limit* $\tilde{\mu}_s = \lim_{T\to\infty} \tilde{\mu}^T_s$ *exists, and* $\nu_s = \Theta\mu_s$ *is a positive Borel measure of finite mass on* $I_N(s)$.

(c) *The functions* $\tilde{\mathcal{I}}_T \circ \Theta^{-1}$ *converge uniformly on* $\mathcal{M}_N$ *to a Hölder continuous function* $\mathcal{I}(\xi^-, s, \xi^+)$. *The function* $\mathcal{I}(\xi^-, s, \xi^+)$ *can be extended to a Hölder continuous function* $\widehat{\mathcal{M}}_N$.

Proof. Since these claims are rather standard we will be brief.

(a) Let

$$\bar{H}_T(\sigma) = \sum_{i=1}^{T} h^+(\tau^i\sigma) + \sum_{i=0}^{T} h^-(\tau^{-i}\sigma)$$

where $h^-(\tau^{-i}\sigma) = \lambda^-(\mathcal{P}(\sigma))$ with

$$\lambda^-(\Psi) = -\log \det(\Lambda_-^{-1}\mathcal{L}^-(X^{-1}(\Psi))). \tag{6.16}$$

Define the measure

$$\bar{\mu}^T(d\sigma) = \frac{1}{\bar{Z}_T} e^{\bar{H}_T(\sigma)} \bar{m}(d\sigma)$$

and its image $\bar{\nu}^T = \Theta_N \bar{\mu}^T$. It is well known that $\bar{\nu} \equiv \lim_{T\to\infty} \bar{\nu}^T$ exists and is absolutely continuous with respect to the Lebesgue measure with a continuous density. Thus, its restriction to $\mathcal{Q}_s$ is given in the $\xi^\pm$ coordinates as

$$\bar{\nu}_s(d\xi^+,\, d\xi^-) = g_s(\xi^+, \xi^-)\, d\xi^+\, d\xi^-$$

with g continuous.

On the other hand, we may decompose $\bar{\nu}$ as we did μ above and get

$$\bar{\nu}_s(d\xi^+,\, d\xi^-) = e^{\bar{\mathcal{I}}(\xi^-, s, \xi^+)} \nu_{s,\xi^-}(d\xi^+) \bar{\nu}_s(d\xi^-)$$

for some Borel measure $\bar{\nu}_s$ and continuous $\bar{\mathcal{I}}$. Hence we conclude that ν_{s,ξ^-} is absolutely continuous with respect to the Lebesgue measure on $J_N(s,\xi^-)$:

$$\nu_{s,\xi^-}(d\xi^+) = f_{s,\xi^-}(\xi^+)\, d\xi^+$$

where $f_{s,\xi^-}(\xi^+)$ is continuous in all variables.

Let now $\mathcal{A}_u$ be the map $\mathcal{A}$ restricted to the unstable manifold. We get then

$$(\mathcal{A}_u \nu_{s,\xi^-})(d\xi_1^+) = \det(\tilde{\mathcal{L}}^+(\Psi(\xi^-, s, \xi^+)))^{-1} f_{s,\xi^-}(\xi^+)\, d\xi_1^+, \tag{6.17}$$

where $\mathcal{A}(\Psi(\xi^-, s, \xi^+)) = \Psi(\xi_1^-, s_1, \xi_1^+)$.

From the definition of $\tilde{\mu}_{s,\sigma^-}$ (6.12), one concludes that $\tau\mu_{s,\sigma^-} = z(s,\sigma^-)e^{h^+}\mu_{s_1,\sigma_1^-}$ where we have set $\tau\sigma = \sigma_1$ and $z(s,\sigma^-) = \lim_{T\to\infty} Z_{T-1}(s,\sigma^-) Z_T^{-1}(s,\sigma^-)$.

From this it follows that

$$(\mathcal{A}_u \nu_{s,\xi^-})(d\xi_1^+) = \tilde{z}(s,\xi^-) \det(\tilde{\mathcal{L}}^+(\Psi(\xi^-, s, \xi^+)))^{-1} f_{s,\xi^-}(\xi_1^+)\, d\xi_1^+ \tag{6.18}$$

so that we have $f_{s_1,\xi_1^-}(\xi_1^+) = \tilde{z}(s,\xi^-) f_{s,\xi^-}(\xi^+)$. We can now fix s, ξ^- and choose s_{-n}, ξ_{-n}^- and $J_{-n} \subset J(s_{-n}, \xi_{-n}^-)$ so that $\mathcal{A}_u^n$ maps J_{-n} bijectively onto $J(s, \xi^-)$. It follows that

$$f_{s,\xi^-}(\xi^+) = \prod_{i=1}^{n} \tilde{z}(s_{-i}, \xi_{-i}^-) f_{s_{-n},\xi_{-n}^-}(\xi_{-n}^+)$$

with $\xi_{-n}^+ \in J_{-n}$. By expansiveness of $\mathcal{A}_u$ the intervals J_{-n} shrink exponentially and the right-hand side of the above equation converges to a ξ^+-independent limit. Clearly this limit is fixed by the fact that ν_{s,ξ^-} is a probability measure.

(b) In statistical mechanics terms $\tilde{\mu}_s$ is the Gibbs measure for spins σ^- in the half-space of negative time, with $s \vee \sigma^+$ as boundary conditions in non-negative times. The $T \to \infty$ limit then follows from exponential decay of interactions guaranteed by the Hölder property of h^+.

(c) We have

$$\mathcal{I}(\xi^-, s, \xi^+) = \lim_{T\to\infty} \sum_{i=0}^{T} \lambda^+(\mathcal{A}^{-i}(\Psi^N(\xi^-, s, \xi^+))) - \lambda^+(\mathcal{A}^{-i}(\Psi^N(\xi^-, s, 0))). \quad (6.19)$$

By Hölder continuity of λ^+ the summand is bounded in absolute value by

$$C(N) d(\mathcal{A}^{-i}(\Psi^N(\xi^-, s, \xi^+)), \mathcal{A}^{-i}(\Psi^N(\xi^-, s, 0))) \le C(N) 2^{-i\eta}.$$

Hence the limit as $T \to \infty$ exists. The extension follows immediately from the representation (6.19). □

To summarize, the SRB measure for $\mathcal{A}_N$ in the $m = (\xi^-, s, \xi^+)$ coordinates can be written as

$$\nu(dm) = e^{\mathcal{I}(m)} 1_{J(s,\xi^-)}(\xi^+) b(ds) \nu_s(d\xi^-) \frac{d\xi^+}{|J(s,\xi^-)|}. \quad (6.20)$$

6.3. *Decomposition in the infinite volume limit.* We are interested in the limit as $N \to \infty$ of the above measures but to study the projected SRB measure we will decompose ν_s, extracting from it a finite-dimensional part ξ_M of the unstable coordinate ξ^+. Thus let us fix an integer M and for $N > M$ write $\mathbb{R}^{\Omega_N} = \mathbb{R}^{\Omega_M} \times \mathbb{R}^{\Omega_N \setminus \Omega_M}$ and $\xi^+ = (\xi_M, \xi^\perp)$ accordingly. The actual value of M we need to study the projected SRB measure will be fixed in the following section. We can rewrite (6.19) as

$$\begin{aligned}\mathcal{I}_N(m) &= \lim_{T\to\infty} \sum_{i=0}^{T} (\lambda^+(\mathcal{A}^{-i}(\Psi^N(\xi^-, s, \xi^+))) - \lambda^+(\mathcal{A}^{-i}(\Psi^N(\xi^-, s, (0, \xi^\perp))))) \\ &\quad + \lim_{T\to\infty} \sum_{i=0}^{T} (\lambda^+(\mathcal{A}^{-i}(\Psi^N(\xi^-, s, (0, \xi^\perp)))) - \lambda^+(\mathcal{A}^{-i}(\Psi^N(\xi^-, s, 0)))) \\ &\equiv \mathcal{J}_N(m) + \mathcal{K}_N(m') \end{aligned} \quad (6.21)$$

where the triple $(\xi^-, s, \xi^\perp)$ was denoted by m'. Let $\mathcal{M}'_N$ be the set of all $m' = (\xi, s, \xi^\perp)$ such that $(\xi, s, \xi^\perp, \xi_M) \in \mathcal{M}_N$ for some ξ_M. Clearly $\mathcal{M}'_N \subset S_N \times C_r^{N,M} = \widehat{\mathcal{M}'_N}$ where $C_r^{N,M}$ is the cube of side r in $\mathbb{R}^{\Omega_N / \Omega_M}$. Given $\xi^\perp \in C_r^{N,M}$, we set

$$\{\xi_M \mid (\xi_M, \xi^\perp) \in J_N(s, \xi^-)\} \equiv J_N(m') \subset C_r^M \quad (6.22)$$

while, given $\xi_M \in C_r^M$, we set

$$\{\xi^\perp \mid (\xi_M, \xi^\perp) \in J_N(s, \xi^-)\} \equiv J_N^\perp(s, \xi^-, \xi_M) \subset C_r^{N,M}. \tag{6.23}$$

Finally, let the projection of the set $J_N(s, \xi^-)$ to the $\xi^\perp$-direction, i.e. to $\mathbb{R}^{\Omega_N \setminus \Omega_M}$, be denoted by $J_N^\perp(s, \xi^-)$. Clearly we have

$$J_N^\perp(s, \xi^-) = \bigcup_{\xi_M \in C_r^M} J_N^\perp(s, \xi^-, \xi_M).$$

We may then rewrite the SRB measure (6.20) as

$$\nu_N(dm) = \rho_N(dm')\vartheta_{m'}^N(d\xi_M), \tag{6.24}$$

where

$$\begin{aligned} \rho_N(dm') &= e^{\mathcal{K}_N(m')} 1_{J_N^\perp(s,\xi^-)}(\xi^\perp) b(ds) \nu_{Ns}(d\xi^-) \frac{d\xi^\perp}{|J_N(s,\xi^-)|} \\ \vartheta_{m'}^N(d\xi_M) &= e^{\mathcal{J}_N(m)} 1_{J_N(m')}(\xi_M)\, d\xi_M. \end{aligned}$$

Clearly for every finite N and every continuous function T_N on $\mathcal{M}_N$ we can write

$$\int T_N(m)\nu_N(dm) = \int \rho_N(dm') \int \vartheta_{m'}^N(d\xi_M) T_N(m).$$

We now want to show that we can take the limit of this identity.

PROPOSITION 11. *There exist a bounded Hölder continuous function $\mathcal{J}$ on $\widehat{\mathcal{M}}$, a Borel measure $\rho(dm')$ of finite mass on $\widehat{\mathcal{M}}'$ and a Borel set $J(m')$ in C_r^M such that given a continuous function T on $\mathcal{M}_\infty$ we have the decomposition*

$$\int T(m)\nu(dm) = \int \rho(dm') \int \vartheta_{m'}(d\xi_M) T(m)$$

where

$$\vartheta_{m'}(d\xi_M) = e^{\mathcal{J}(m)} 1_{J(m')}(\xi_M)\, d\xi_M.$$

Proof. We show first that the functions $\mathcal{J}_N$ converge to a bounded Hölder continuous function on $\widehat{M}$. For this observe that

$$\begin{aligned} &\lambda^+(\Psi^N(\xi^-, s, \xi^+)) - \lambda^+(\Psi^N(\xi^-, s, (0, \xi^\perp))) \\ &\quad = \sum_{\mathbf{i} \in \Omega_N} (\lambda(\tau_{\mathbf{i}} \Psi^N(\xi^-, s, \xi^+)) - \lambda(\tau_{\mathbf{i}} \Psi^N(\xi^-, s, (0, \xi^\perp)))). \end{aligned} \tag{6.25}$$

By the Hölder continuity of λ (5.8) we have

$$\begin{aligned} &|\lambda(\tau_{\mathbf{i}} \Psi^N(\xi^-, s, \xi^+)) - \lambda(\tau_{\mathbf{i}} \Psi^N(\xi^-, s, (0, \xi^\perp)))| \\ &\quad \le C\varepsilon d(\tau_{\mathbf{i}} \Psi^N(\xi^-, s, \xi^+), \tau_{\mathbf{i}} \Psi^N(\xi^-, s, (0, \xi^\perp)))^\gamma \\ &\quad \le C\varepsilon \left(\sum_{\mathbf{j}} 2^{-|\mathbf{j}|} |\Psi_{\mathbf{j}-\mathbf{i}}^N(\xi^-, s, \xi^+) - \Psi_{\mathbf{j}-\mathbf{i}}^N(\xi^-, s, (0, \xi^\perp))| \right)^\gamma. \end{aligned}$$

From the regularity property of Ψ^N at fixed (s, ξ^-) we infer

$$|\Psi^N_{\mathbf{k}}(\xi^-, s, \xi^+) - \Psi^N_{\mathbf{k}}(\xi^-, s, (0, \xi^\perp))| \leq Ce^{-c\,\mathrm{dist}(\mathbf{k}, \Omega_M)} \tag{6.26}$$

so that

$$|\lambda^+(\Psi^N(\xi^-, s, \xi^+)) - \lambda^+(\Psi^N(\xi^-, s, (0, \xi^\perp)))| \leq C_M \varepsilon \tag{6.27}$$

uniformly in N. Observe finally that from (6.27) we get

$$|\lambda^+(\mathcal{A}^{-i}(\Psi^N(\xi^-, s, \xi^+))) - \lambda^+(\mathcal{A}^{-i}(\Psi^N(\xi^-, s, (0, \xi^\perp))))| \leq C_M e^{-ci} \varepsilon \tag{6.28}$$

because $\Psi^N(\xi^-, s, (0, \xi^\perp))$ and $\Psi^N(\xi^-, s, \xi^+)$ are on the same leaf of the unstable foliation. Convergence follows from convergence of λ, $\mathcal{A}$ and Ψ^N. □

We will next prove that the masses of the measures ρ_N are uniformly bounded, i.e. that $\rho_N(\mathcal{M}'_N) < C$ with C independent from N.

The set $J_N(s, \xi^-)$ can be written as

$$J_N(s, \xi^-) = \bigcap_{\mathbf{i} \in \Omega_N} J_{N\mathbf{i}}(s, \xi^-)$$

where

$$J_{N\mathbf{i}}(s, \xi^-) = \{\xi^+ \mid Y^N_{\mathbf{i}}(m) \in U_{s_{\mathbf{i}}}\}$$

where U_s is the interval spanning the unstable side of the rectangle Q_s of the Markov partition of the linear map A. Moreover $Y^N(m) = X^{-1}(\Psi^N(\xi^-, s, \xi^+))$ is a Hölder continuous function such that $|\delta_{\xi_M} Y^N_{\mathbf{i}}(m)| \leq Ce^{-\beta|\mathbf{i}|}$.

Let us define the functions $(R^\pm(\xi^\perp))_{\mathbf{i}} = C^\pm_{\mathbf{i}} \xi^\perp_{\mathbf{i}}$ where the $C^\pm_{\mathbf{i}} = 1 \pm Ce^{-\omega|\mathbf{i}|}$ for suitable C and ω. We can then define the two sets

$$K^\pm_N(s, \xi^-) = R^\pm(J^\perp_N(s, \xi^-, 0)).$$

From the property of the function Y it follows that, for suitable C and ω we have

$$\begin{aligned} J^\perp_N(s, \xi^-, \xi_M) &\subset K^+_N(s, \xi^-) \quad \text{for every } \xi_M \in C^M_r, \\ J^\perp_N(s, \xi^-, \xi_M) &\supset K^-_N(s, \xi^-) \quad \text{for every } \xi_M \in C^M_{r/2}. \end{aligned}$$

From this it follows that

$$\begin{aligned} &\int e^{\mathcal{K}_N(m')} 1_{J^\perp_N(s,\xi^-)}(\xi^\perp)\, d\xi^\perp \leq \int e^{\mathcal{K}_N(m')} 1_{K^+_N(s,\xi^-)}(\xi^\perp)\, d\xi^\perp \\ &\int e^{\mathcal{I}_N(m)} 1_{J_N(s,\xi^-)}(\xi^+)\, d\xi^+ \geq e^{-C\mathcal{J}} \int e^{\mathcal{K}_N(m')} 1_{K^-_N(s,\xi^-)}(\xi^-) 1_{C^M_{r/2}}(\xi_M)\, d\xi^\perp\, d\xi_M. \end{aligned}$$

The following lemma will allow us to compare the right-hand sides of the two above inequalities.

LEMMA. *For $\xi^\perp \in C^{N,M}_r$ we have $|\mathcal{K}_N(s, \xi^-, R^\pm(\xi^\perp)) - \mathcal{K}_N(s, \xi^-, \xi^\perp)| \leq C$.*

Proof. $\mathcal{K}_N(s, \xi^-, \xi^\perp)$ is given by (6.21). We can write

$$\mathcal{K}_N(s, \xi^-, \xi^\perp) = \sum_i \mathcal{K}_{Ni}(s, \xi^-, \xi^\perp).$$

We start bounding the term with $i = 0$. We have

$$\begin{aligned}
&|\mathcal{K}_{N0}(s, \xi^-, R^\pm(\xi^\perp)) - \mathcal{K}_{N0}(s, \xi^-, \xi^\perp)| \\
&\quad \le |\lambda^+(\Psi^N(\xi^-, s, (0, R^\pm(\xi^\perp)))) - \lambda^+(\Psi^N(\xi^-, s, (0, \xi^\perp)))| \\
&\quad \le \sum_{\mathbf{i}} |\lambda(\tau_{\mathbf{i}}\Psi^N(\xi^-, s, (0, R^\pm(\xi^\perp)))) - \lambda(\tau_{\mathbf{i}}\Psi^N(\xi^-, s, (0, \xi^\perp)))|. \qquad (6.29)
\end{aligned}$$

We may now proceed as after (6.25), replacing (6.26) by

$$|\Psi^N_{\mathbf{k}}(\xi^-, s, (0, R^\pm(\xi^\perp))) - \Psi^N_{\mathbf{k}}(\xi^-, s, (0, \xi^\perp))| \le Ce^{-c\,\mathrm{dist}(\mathbf{k},\Omega_M)}e^{-\omega|\mathbf{i}|}$$

and bounding (6.29) by $C\varepsilon$. As in (6.28) we obtain that

$$|\mathcal{K}_{Ni}(s, \xi^-, R^\pm(\xi^\perp)) - \mathcal{K}_{Ni}(s, \xi^-, \xi^\perp)| \le Ce^{-c|i|}\varepsilon,$$

which yields the claim. □

Using the above lemma it follows that

$$\frac{\int e^{\mathcal{K}_N(m')} 1_{J_N^\perp(s,\xi^-)}(\xi^\perp)\, d\xi^\perp}{\int e^{\mathcal{I}_N(m)} 1_{J_N(s,\xi^-)}(\xi^+)\, d\xi^+} \le C.$$

The boundedness of the measure $\rho_N(dm')$ follows from the above estimate and the fact that $\nu(dm)$ is a probability measure.

Let $T(m)$ now be a continuous function on $\hat{\mathcal{M}}$. For $m \in \hat{\mathcal{M}}$ define $P_N m \in \hat{\mathcal{M}}$ to be the point that coincides with m on Ω_N and is extended periodically outside Ω_N, i.e. $P_N m$ is also in $\hat{\mathcal{M}}_N$; see comment before (2.4). Set $T_N(m) = T(P_N m)$. The continuity of T and the weak convergence of ν_N imply

$$\int T(m)\nu(dm) = \lim_{N\to\infty} \int T_N(m)\nu_N(dm).$$

Decomposing as in §6.2, we get

$$\int T(m)\nu(dm) = \lim_{N\to\infty} \int b_N(ds)\nu_{Ns}(d\xi^-) \int \nu_{N,s,\xi^-}(d\xi^+)\widetilde{T}_N(m)$$

with

$$\widetilde{T}(m) = e^{\mathcal{I}(m)}T(m)$$

and

$$\nu_{N,s,\xi^-} = e^{\mathcal{I}_N(m)} 1_{J_N(s,\xi^-)}(\xi^+) \frac{d\xi^+}{|J_N(s, \xi^-)|}.$$

By the weak convergence of both measures,

$$\int T(m)\nu(dm) = \int b(ds)\nu_s(d\xi^-) \lim_{N\to\infty} \int \nu_{N,s,\xi^-}(d\xi^+)\widetilde{T}_N(m).$$

We can rewrite the last limit as

$$\lim_{N\to\infty} \int \nu_{N,s,\xi^-}(d\xi^+)\widetilde{T}_N = \lim_{N\to\infty} \int \rho_{N,s,\xi^-}(d\xi^\perp) g_N(m')$$

where

$$g_N(m') = \int 1_{J_N(m')}(\xi_M)\widetilde{T}_N(m)\,d\xi_M.$$

Now let

$$J^K(m') = \bigcap_{N>K} \bigcap_{\mathbf{i}\in\Omega_N/\Omega_K} \bigcap_{\xi_\mathbf{i}} J_N(\bar{m}'); \tag{6.30}$$

i.e. we take the union over $N \geq K$ and $\bar{m}'$ such that $P_K\bar{m}' = P_K m'$. Note that $J^K(m')$ depends on m' only through $P_K m$. Set

$$g^K(m') = \int 1_{J^K(m')}(\xi_M)\widetilde{T}_K(m)\,d\xi_M.$$

By compactness there is a subsequence of the measures ρ_{N,s,ξ^-} that converges weakly to some ρ_{s,ξ^-}. Moreover, $g^K(m')$ are bounded measurable functions in C_r^K, hence they can be approximated by continuous ones on sets whose complements have arbitrary small Lebesgue measure and thus arbitrarily small ρ_{N,s,ξ^-} measure, uniformly in N. Hence we get the limit

$$\lim_{i\to\infty}\int \rho_{N_i,s,\xi^-}(d\xi^+)g^K(m') = \int \rho_{s,\xi^-}(d\xi^+)g^K(m').$$

We need to estimate

$$\begin{aligned}\int \rho_{N,s,\xi^-}&(d\xi^\perp)((g_N(m') - g^K(m')))\\ &= \int \rho_{N,s,\xi^-}(d\xi^\perp)\cdot\int(1_{J_N(m')}(\xi_M)\widetilde{T}_N(m) - 1_{J^K(m')}(\xi_M)\widetilde{T}^K(m))\,d\xi_M.\end{aligned}$$

By continuity of T, we have that $\|\widetilde{T}_N - \widetilde{T}^K\|_\infty \to 0$ as $N, K \to \infty$. Thus it suffices to show

$$\int \rho_{N,s,\xi^-}(d\xi^\perp)\int(1_{J_N(m')}(\xi_M) - 1_{J^K(m')}(\xi_M))\widetilde{T}^K(m)\,d\xi_M \tag{6.31}$$

tends to zero as $N, K \to \infty$. Since $J_N(m') \subset J^K(m')$ the difference between the characteristic functions is non-zero only if there exists a $\mathbf{j} \in \Omega_N$ and m', $\bar{m}'$, with $P_K\bar{m}' = P_K m'$, such that

$$\begin{aligned}Y_j(P_N m', \xi_M) &\notin U_{s_\mathbf{j}},\\ Y_j(\bar{m}', \xi_M) &\in U_{s_\mathbf{j}}.\end{aligned}$$

Moreover, on the support of ρ_{N,s,ξ^-} we have

$$Y_\mathbf{j}(P_N m', \tilde{\xi}_M) \in U_{s_\mathbf{j}}$$

for some $\tilde{\xi}_M$. Recall that $Y_\mathbf{j}(m) = \xi_\mathbf{j}^+ + \epsilon_\mathbf{j}(m)$ and for $K \geq |\mathbf{j}|$ we have

$$|\epsilon_\mathbf{j}(P_N m', \xi_M) - \epsilon_\mathbf{j}(\bar{m}', \xi_M)| \leq C\varepsilon e^{-c(K-|\mathbf{j}|)}$$

and for $|\mathbf{j}| \geq M$

$$|\epsilon_\mathbf{j}(P_N m', \xi_M) - \epsilon_\mathbf{j}(P_N m', \tilde{\xi}_M)| \leq C\varepsilon e^{-c(|\mathbf{j}|-M)}.$$

Thus we may conclude that (6.31) is bounded by Ce^{-cK} and therefore

$$\lim_{N\to\infty}\int \rho_{N,s,\xi^-}(d\xi^{\perp})g_N(m') = \lim_{K\to\infty}\int \rho_{s,\xi^-}(d\xi^{+})g^K(m').$$

Since $P_{K+1}\bar{m}' = P_{K+1}m'$ implies $P_K\bar{m}' = P_K m'$ we get $J^{K+1}(m') \subset J^K(m')$. Defining the measurable set

$$J(m') = \bigcap_K J^K(m')$$

we get by dominated convergence

$$\lim_{K\to\infty} g^K(m') = \int 1_{J(m')}(\xi_M)\widetilde{T}(m)\, d\xi_M$$

whereby the proof is completed. □

7. *The projected SRB measure*

We now turn to the study of the projected SRB measure and to the proof of Proposition 1 and Theorem 2. We work with general $N \leq \infty$ and suppress the N-dependence if no confusion can arise.

Recall that $\Psi_0 : \mathcal{M} \to \mathbb{T}$ is the projection to the torus at the origin of Ω expressed in the (ξ^-, s, ξ^+) coordinate representation for Ψ. Ψ_0 is continuous on $\mathcal{M}$ and for fixed s, ξ^- is real-analytic in ξ^+. Let $T(\psi)$ be a continuous function from $\mathbb{T}$ to $\mathbb{R}$ and $M < N$ to be fixed later. By definition of the projection and Proposition 11,

$$\int_{\mathbb{T}} T(\psi)\mathbf{P}^*\mu(d\psi) = \int T(\Psi_0)\mu(d\Psi) = \int \rho(dm')\int d\xi_M\, T(\Psi_0(m',\xi_M))a(m',\xi_M) \tag{7.1}$$

where we set

$$a(m',\xi_M) = e^{\mathcal{J}(m)}1_{J(m')}(\xi_M). \tag{7.2}$$

Let $\omega_{m'}(d\psi)$ be the image under Ψ_0 of the measure $l(m',\xi_M) = a(m',\xi_M)\,d\xi_M$, i.e.

$$\omega_{m'}(A) = l(\Psi_0(m',\cdot)^{-1}(A)).$$

Then (7.1) may be written as

$$\int_{\mathbb{T}} T(\psi)\mathbf{P}^*\mu(d\psi) = \int \rho(dm')\int_{\mathbb{T}}\omega_{m'}(d\psi)T(\psi) \tag{7.3}$$

and we need to study next under what conditions the measure $\omega_{m'}$ is absolutely continuous with respect to the Lebesgue measure on the torus $\mathbb{T}$.

In the $\pm$ coordinates of $\mathbb{T}$ we have $\Psi_0 = (\psi^+,\psi^-)$ with $\psi^+ = \xi_0 + \mathcal{O}(\epsilon)$. It will be convenient to change coordinates on $\mathcal{M}$ by solving ξ_0 in terms of ψ^+. Thus write $\xi_M = (\xi_0,\xi)$ and let $f_{m'\xi}$ be the inverse of $\xi_0 \to \psi^+(m',\xi_0,\xi)$. Then the map $\Psi\circ f_{m'\xi}$ provides coordinates (m',ξ,ψ^+) on $\mathcal{M}$ and in particular we get for $\phi = \Psi_0\circ f_{m'\xi}$

$$\phi(m',\xi,\psi^+) = (\psi^+,\psi^-(m',\xi,\psi^+)) \tag{7.4}$$

where ψ^- is continuous in m' and real-analytic in ξ,ψ^+. The measure $\omega_{m'}$ is the image of $a\circ f_{m'\xi}\, d\psi^+\, d\xi$ under the map ϕ. Our objective is to show that provided a non-degeneracy condition is satisfied $\omega_{m'}$ is absolutely continuous with respect to the Lebesgue measure $d\psi^+ \, d\psi^-$. Since a is bounded it suffices to show $\phi(d\psi^+ \, d\xi)$ is absolutely continuous.

Clearly, the absolute continuity fails if the function ψ^- in (7.4) is constant in ξ. This turns out to be both a necessary and a sufficient condition, as we will now set out to prove.

Let $\xi' = (\xi^\perp, \xi)$ so that $m \in \mathcal{M}$ is given by $m = (\xi^-, s, \xi', \psi^+)$. For a multi-index $\mathbf{n} = (n_\mathbf{i})_{\mathbf{i} \in \Omega \setminus \mathbf{o}}$ denote by $|\mathbf{n}| := \sum |n_\mathbf{i}|$ and by supp $\mathbf{n}$ the set of $\mathbf{i}$ such that $n_\mathbf{i} \neq 0$.

PROPOSITION 12. *Suppose that for some $m \in \mathcal{M}$, there exist integer $k \geq 0$ and a multi-index $\mathbf{n} \neq 0$ such that*

$$\partial_{\psi^+}^k \partial_\xi^\mathbf{n} \phi(m) \neq \mathbf{o}. \tag{7.5}$$

Then for every $m \in \mathcal{M}$ there exists $k(m) \geq 0$ and $\mathbf{n}(m) \neq 0$ such that (7.5) holds. Moreover there exists an integer M such that we may assume supp $\mathbf{n}(m) \subset \Omega_M$ *and $|\mathbf{n}(m)| < M$ for all m.*

Proof. Suppose for some m no such k and $\mathbf{n}$ exist. By real analyticity of ϕ in ψ^+ and ξ this means $\phi(\xi^-, s, \xi, \psi^+)$ is constant in ξ for all ψ. Going back to the coordinates (ξ^-, s, ξ^+) we infer that the rank of the map $D_{\xi^+}\Psi_0(\xi^-, s, \xi^+)$ is one for all ξ^+ on the domain. Since the map $\xi^+ \to \Psi_0(\xi^-, s, \xi^+)$ equals the projection $\mathbf{P}$ to the origin applied to the embedding $S^+_{\Psi'}$ given by Proposition 4, with $\Psi' = \Psi(\xi^-, s, 0)$, it follows that the rank of $D_{\xi^+}\mathbf{P}S^+_{\Psi'}(\xi^+)$ equals one for all $\xi^+ \in \mathbb{R}^\Omega$. But the image of $\mathbb{R}^\Omega$ under $S^+_{\Psi'}$ is dense in $\mathcal{T}$ so by continuity the rank equals one for all $\Psi \in \mathcal{T}$. This in turn implies that $\phi(\xi^-, s, \xi, \psi^+)$ is constant in ξ for all s, ξ^-, ψ, i.e. the condition (7.5) holds nowhere. This takes care of the first claim.

The second claim is non-vacuous only for $N = \infty$. Thus suppose for all $m \in \mathcal{M}$, $k(m) \geq 0$ and $\mathbf{n}(m) \neq 0$ exist such that (7.5) holds. By continuity it holds in a neighborhood of m with the same $k(m)$ and $\mathbf{n}(m)$ and thus by compactness of $\mathcal{M}$ we infer the existence of $M < \infty$. □

We continue now the study of the measure $\omega_{m'}$ supposing the condition (7.5) holds. We choose the M in (7.1) as in Proposition 12. Given a point $m = (m', \tilde{\xi}, \tilde{\psi}^+)$ let us fix $k(m)$ to be the smallest of the k satisfying (7.5). Then we may write, for (ξ, ψ^+) in some neighborhood $U(m)$ of the origin,

$$\psi^-(m', \tilde{\xi} + \xi, \tilde{\psi}^+ + \psi^+) - \psi^-(m', \tilde{\xi}, \tilde{\psi}^+) = (\psi^+)^{k(m)} f(m', \xi, \psi^+)$$

with f real-analytic in (ξ, ψ^+) and $f(m', \xi, 0)$ a non-constant function.

LEMMA 4. *There exist a neighborhood $V(m)$ of the origin in $\mathbb{R}^{\Omega_M}$ such that the image of the Lebesgue measure under the map $F : V(m) \to \mathbb{R}^2$ given by $(\xi, \psi^+) \to (\psi^+, (\psi^+)^k f(m', \xi, \psi^+))$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^2$.*

Proof. See Appendix A. □

By compactness we may cover $J(m')$ by a finite number of such neighborhoods and conclude the absolute continuity of $\omega_{m'}$ for each m':

$$\omega_{m'}(d\psi) = \omega_{m'}(\psi) \, d\psi$$

with $\omega_{m'}(\psi)$ non-negative and integrable. Thus (7.3) becomes

$$\int_{\mathbb{T}} T(\psi)\mathbf{P}^*\mu(d\psi) = \int \rho(dm') \int_{\mathbb{T}} \omega_{m'}(\psi)T(\psi)\,d\psi. \tag{7.6}$$

Since, by construction, $\int \omega_{m'}(\psi)\,d\psi \leq C$ for all m' we can conclude, by the Fubini–Tonelli theorem, that $\mathbf{P}^*\mu(d\psi) = \eta(\psi)\,d\psi$ with $\eta(\psi) = \int \rho(dm')\omega_{m'}(\psi)$ in $L^1(\mathbb{T})$.

We will now turn to the proof of Proposition 1, i.e. we will characterize the systems for which the projection is singular.

LEMMA 5. *Suppose (7.5) is violated. Then the unstable manifold is a product of curves*

$$W^+(\Psi) = \times_{\mathbf{i}\in\Omega}\, \gamma_{\mathbf{i}}(\Psi)$$

where $\gamma_{\mathbf{i}}(\Psi) : \mathbb{R} \to \mathbb{T}_{\mathbf{i}}$ *is an embedding to the torus at* $i \in \Omega$.

Proof. From the proof of Proposition 12 we know that the map $D\mathbf{P}S^+_\Psi(\xi)$ has rank 1 for every $\xi \in \mathbb{R}^{\Omega_N}$. Thus the vectors $v_{\mathbf{i}}(\Psi,\xi) = \partial_{\xi_{\mathbf{i}}}\mathbf{P}S^+_\Psi(\xi) \in \mathbb{R}^2$ are parallel. Since $v_{\mathbf{o}} = (0,1) + \mathcal{O}(\epsilon_0) \neq 0$ there exist functions $\lambda_{\mathbf{i}}(\Psi,\xi)$, real-analytic in $\xi \in U$, such that

$$v_{\mathbf{i}} = \lambda_{\mathbf{i}} v_{\mathbf{o}}. \tag{7.7}$$

Let $\mathbf{P}^+$ be the orthogonal projection in $\mathbb{R}^{2\Omega_N}$ to the unstable space E^+ of $\mathcal{A}_0$ and let

$$f_\Psi = \mathbf{P}^+ S^+_\Psi.$$

Since S^+_Ψ is a real-analytic embedding in $\mathbb{R}^{2\Omega_N}$ and $\mathbf{P}^+$ is one-to-one on the image of S^+_Ψ we conclude that f_Ψ is a real-analytic diffeomorphism of $\mathbb{R}^{\Omega_N}$. Let us change the parameterization of $W^+(\Psi)$ using f_Ψ, i.e. let $\tilde{S}^+_\Psi = S^+_\Psi \circ f_\Psi^{-1}$ and $\tilde{v}_{\mathbf{i}} = \partial_{\xi_{\mathbf{i}}}\mathbf{P}\tilde{S}^+_\Psi$. Then

$$\mathbf{P}^+\tilde{S}^+_\Psi(\xi) = \xi$$

and hence $\mathbf{P}^+\tilde{v}_{\mathbf{i}} = \delta_{\mathbf{io}}$. On the other hand, by (7.7)

$$\tilde{v}_{\mathbf{i}} = \partial_{\xi_{\mathbf{i}}}\mathbf{P}\tilde{S}^+_\Psi = v_{\mathbf{o}} \circ f_\Psi^{-1} \sum_{\mathbf{j}} \lambda_{\mathbf{j}} \circ f_\Psi^{-1} \partial_{\xi_{\mathbf{i}}} f^{-1}_{\Psi_{\mathbf{j}}}.$$

Thus, combining these identities with $\mathbf{P}^+ v_{\mathbf{o}} \neq 0$, we infer that $\sum_{\mathbf{j}} \lambda_{\mathbf{j}} \circ f^{-1}\partial_{\xi_{\mathbf{i}}} f_{\mathbf{j}}^{-1} = 0$. Therefore $\tilde{v}_{\mathbf{i}}$ vanishes identically for $\mathbf{i} \neq \mathbf{o}$. Hence $\mathbf{P}\tilde{S}^+_\psi(\xi)$ depends on ξ only through $\xi_{\mathbf{o}}$.

Let $\tau_{\mathbf{i}}$ for $\mathbf{i} \in \Omega_N$ be the translation $(\tau_{\mathbf{i}}\Psi)_{\mathbf{j}} = \Psi_{\mathbf{i}+\mathbf{j}}$ and on ξ similarly. Then $\mathbf{P}_i\tilde{S}^+_\Psi(\xi) = \mathbf{P}\tilde{S}^+_{\tau_{\mathbf{i}}\Psi}(\tau_{\mathbf{i}}\xi)$. Therefore, $\mathbf{P}_{\mathbf{i}}\tilde{S}^+_\Psi(\xi) = \gamma_{\mathbf{i}}(\Psi,\xi_{\mathbf{i}})$ for a $\gamma_{\mathbf{i}}$ satisfying the claim of the lemma. □

Denote by $\mathcal{O} = X(0)$ the fixed point of $\mathcal{A}$. Observe that, due to the periodic boundary conditions, all components of $\mathcal{O}$ are equal to the same value $\psi_{\mathcal{O}}$. Thus all the curves $\gamma_i(\mathcal{O})$ are identical. Since the restriction of $\mathcal{A}$ to the Ω_M-periodic points of $\mathcal{T}$ is $\mathcal{A}_M$ we may infer that $\times_\Omega W_1^+(\psi_{\mathcal{O}}) \subset W^+(\mathcal{O})$. Thus $\gamma_i(\mathcal{O}) = W_1^+(\psi_{\mathcal{O}})$ and we have obtained

$$W^+(\mathcal{O}) = \times_\Omega W_1^+(\psi_{\mathcal{O}}). \tag{7.8}$$

Let $\widetilde{\mathcal{A}} = \tilde{X}^{-1}\mathcal{A}\widetilde{X}$ where $\widetilde{X} = \times_\Omega X_1$. Then (7.8) implies $\widetilde{W}^+(0) = \times_{\mathbf{i}} W_A^+(0)$ where $\widetilde{W}^+(\Psi)$ and $W_A^+(\psi)$ are the unstable manifolds of the map $\widetilde{\mathcal{A}}$ and of the linear torus map A.

Due to the density of $W_A^+(0)$, we get that for any $\Psi \in \mathcal{T}$

$$\widetilde{W}^+(\Psi) = \times_\Omega \, W_A^+(\Psi_{\mathbf{i}}). \tag{7.9}$$

Indeed given $\Psi \in \mathcal{T}$ we can always find a sequence of points $\Psi_n \in \widetilde{W}^+(0)$ such that $\lim_{n\to\infty} \Psi_n = \Psi$. Observe that $\widetilde{W}^+(\Psi_n) = \times_\Omega \, W_A^+((\Psi_n)_{\mathbf{i}})$ because $\widetilde{W}^+(\Psi_n) = \widetilde{W}^+(0)$. Let now $\widetilde{W}_r^+(\Psi_n)$ be the sphere of radius r and center Ψ_n in $\widetilde{W}^+(\Psi_n)$. Due to the continuity of the unstable foliation, it follows that, for every positive r, $\widetilde{W}_r^+(\Psi_n)$ converges to $\widetilde{W}_r^+(\Psi)$. This proves (7.9).

Observe now that, for every $\Psi = (c_{\mathbf{i}} e_{\mathbf{i}}^+)_{\mathbf{i}\in\Omega} \in \widetilde{W}^+(0)$, we have that $\widetilde{\mathcal{A}}(\Psi) = (c'_{\mathbf{i}} e_{\mathbf{i}}^+)_{\mathbf{i}\in\Omega}$, where we can write $c'_{\mathbf{i}} = \lambda^+ c_{\mathbf{i}} + f_{\mathbf{i}}(\Psi)$ with f defined and continuous on $\widetilde{W}^+(0)$. If $\Psi \notin \widetilde{W}^+(0)$ we can again approximate it by a sequence Ψ_n. The continuity of the map $\widetilde{\mathcal{A}}$ implies that the limit of $f(\Psi_n)$ exists and is independent of the chosen sequence. Finally, we obtain

$$(\widetilde{\mathcal{A}}\Psi)_{\mathbf{i}} = A\Psi_{\mathbf{i}} + f_{\mathbf{i}}(\Psi)e_+,$$

which proves our proposition. □

8. *Perturbative characterization of singular couplings*

Proposition 1 gives a geometric characterization of the singular couplings. This characterization, however, is not directly testable for a given interaction $\mathcal{F}$. We want to discuss here a more practical although less general way to decide whether a given interaction $\mathcal{F}$ is singular. For this purpose we will write $\mathcal{F} = \varepsilon\mathcal{G}$ with $\mathcal{G} = O(1)$ and ε small. From Proposition 12 and its proof we get immediately the following lemma.

LEMMA 6. *Given $\mathcal{G}$ if* $\mathrm{rank}(\partial_\varepsilon D\mathbf{P}S_\Psi^+(0)|_{\varepsilon=0}) \not\equiv 1$ *then there exists ε_0, depending on $\mathcal{G}$ but not on N, such that for all $\varepsilon \le \varepsilon_0$ the coupled system $\mathcal{A}_N$, given by (2.1) and (2.2) with $\mathcal{F} = \varepsilon\mathcal{G}$, is non-degenerate.*

It is rather easy to compute explicitly $\partial_\varepsilon D\mathbf{P}S_\Psi^+(0)|_{\varepsilon=0}$.

LEMMA 7. *If* $\mathrm{rank}(\partial_\varepsilon D\mathbf{P}S_\Psi^+(0)|_{\varepsilon=0}) \equiv 1$ *then* $\partial_{e_{\mathbf{i}}^+} f^-(\Psi) \equiv 0$.

Proof. Observe that the first order in ε of the matrix $\partial_{\xi_{\mathbf{i}}}\partial_{\xi_{\mathbf{j}}}\mathbf{P}S_\Psi^+(0)$ is the 2×2 matrix obtained by selecting in the $2\Omega \times \Omega$ matrix $\chi^+(\Psi)$, see §4, the rows relative to the $+$ and $-$ directions of $\Psi_{\mathbf{o}}$ and the $\mathbf{i},\mathbf{j}$ columns. If $\mathrm{rank}(\partial_\varepsilon D\mathbf{P}S_\Psi^+(0)|_{\varepsilon=0}) = 1$ then for every $\mathbf{i}$ and $\mathbf{j}$ we have $\det(\partial_\varepsilon\partial_{\xi_{\mathbf{i}}}\partial_{\xi_{\mathbf{j}}}\mathbf{P}S_\Psi^+(0)|_{\varepsilon=0}) = 0$. By the choice of the $++$ part of the matrix $\chi^+(\Psi)$, see comment before (4.12), we get that, to the first order, $\det\partial_{\xi_{\mathbf{i}}}\partial_{\xi_{\mathbf{j}}}\mathbf{P}S_\Psi^+(0) = 0$ for every f unless $\mathbf{i} = \mathbf{o}$ or $\mathbf{j} = \mathbf{o}$.

Expanding (4.17) at first order in ε we get that

$$\det(\partial_\varepsilon\partial_{\xi_{\mathbf{i}}}\partial_{\xi_{\mathbf{o}}}\mathbf{P}S_\Psi^+(0)|_{\varepsilon=0}) = (\mathbf{T}_1^{-1}\partial_{e_{\mathbf{i}}^+} f^-)(\Psi).$$

But $\mathbf{T}_1^{-1}$ is a bounded linear operator, see §4, so that we must have $\partial_{e_{\mathbf{i}}^+} f^-(\Psi) \equiv 0$, which proves the lemma. □

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A. *Appendix*

Proof of Lemma 4. Suppressing the m'-dependence and denoting ψ^+ by ψ, we need to study the image η of Lebesgue measure under the map

$$F(\xi, \psi) = (\psi, \psi^k f(\xi, \psi))$$

in some neighborhood U of the origin of $\mathbb{R}^d \times \mathbb{R}$. By assumption, we may write for some n

$$f(z, 0) = \sum_{|\alpha|=n} a_\alpha z^\alpha + \mathcal{O}(|z|^{n+1})$$

where not all a_α vanish. Thus $h(z) := \sum_{|\alpha|=n} \tilde{a}_\alpha z^\alpha$ is a homogeneous polynomial of degree n that does not vanish identically and so there exists $v \in \mathbb{R}^d$, $|v| = 1$, such that $h(v) \neq 0$. Choosing an orthogonal matrix $\mathcal{O}$ such that $\mathcal{O}e_d = v$ where $e_d = (0, \ldots, 1)$ we see that we may assume without loss that $a_{(0,\ldots,n)} \neq 0$. Writing $z = (u_1, \ldots, u_{d-1}, s)$ and defining the function

$$g(\psi, u, s) := \partial_s f(z, \psi)$$

we may write

$$g(\psi, u, s) = \sum_r b_r(\psi, u) s^r$$

so that in a neighborhood V of the origin of $\mathbb{R}^{d-1} \times \mathbb{R}$ there exists a constant B such that

$$|b_r(\psi, u)| \leq B^r.$$

Moreover

$$b_n(0, 0) = \gamma \neq 0,$$
$$b_r(0, 0) = 0, \quad \text{for } r < n.$$

Choose $\rho > 0$ such that

$$|b_{n+1}(0, 0)s + b_{n+2}(0, 0)s^2 + \cdots| < \gamma/2$$

for $|s| \leq \rho$. Moreover let $\mathcal{D} = \{s \in \mathbb{C} \mid |s| < \rho\}$. Then the holomorphic function $g(0, 0, s)$ has an n-fold zero at 0 and no other zeros in $\mathcal{D}$. Furthermore

$$|g(0, 0, s)| \geq \frac{|\gamma|}{2} \rho^n$$

for $|s| = \rho$. By continuity there is a neighborhood U of zero in $\mathbb{R}^{d-1} \times \mathbb{R}$ such that for $(u, \psi) \in U$

$$|g(u, \psi, s)| \geq \frac{|\gamma|}{4} \rho^n$$

for $|s| = \rho$. By Rouché's theorem $g(u, \psi, s)$ has exactly n zeros in $\mathcal{D}$ (counted with multiplicity) when $(u, \psi) \in U$.

Fix $(u, \psi) \in U$ and let $s_1, \ldots, s_m$ be the zeros of $g(u, \psi, s)$ with multiplicities $n_1, \ldots, n_m$ in $\mathcal{D}$. Then $\prod_i |(s - s_i)^{n_i}| \leq (2\rho)^n$ for $|s| \leq \rho$. Therefore,

$$\phi(s) = \frac{g(u, \psi, s)}{\prod_i (s - s_i)^{n_i}}$$

is analytic in $\mathcal{D}$, has no zero in $\mathcal{D}$, and is bounded in absolute value from below by

$$\frac{|\gamma|}{4}\rho^n \Big/ (2\rho)^n = \frac{|\gamma|}{2^{n+2}}$$

on $\partial\mathcal{D}$. By the maximum principle

$$\phi(s) \geq \frac{|\gamma|}{2^{n+2}}$$

for all $s \in \mathcal{D}$.

Fix now $\psi^+ \neq 0$. From the preceding discussion we infer that the function $s \to (\psi^+)^k f((u,s),\psi^+)$ has $m(\psi^+) \leq n$ critical points $s_i(\psi^+)$ and therefore $k \leq m(\psi^+)$ critical values ψ_i^-. The function

$$\eta_u(\psi^+,\psi^-) = \int ds\, \delta(\psi^- - (\psi^+)^k f((u,s),\psi^+))$$

is smooth in the complement of these critical values. Let $\psi^- \in U_i \setminus \psi_i^-$ where U_i is a small enough neighborhood of ψ_i^-. Let s_j be a critical point giving rise to the critical value ψ_i^-. Integrating over a small neighborhood V_j of s_j we get

$$\int_{V_j} ds\, \delta(\psi^- - (\psi^+)^k f((u,s),\psi^+)) = \int_{V_j} ds\, \delta(\psi^- - \psi_i^- - (\psi^+)^k \alpha_j(s)(s-s_j)^{n_j})$$

where $\alpha_j(s)$ is bounded away from zero in V_j. Performing the integration we obtain

$$\eta_u(\psi^+,\psi^-) = \sum_j a_j(\psi^-,\psi^+,u)(\psi^- - \psi_i^-)^{(1/n_j)-1}$$

where a_j is bounded in $\psi^- \in U_i$ and the sum runs over the critical points s_j giving rise to the critical value ψ_i. Hence, for each $\psi^+ \neq 0$, $\eta_u(\psi^+,\psi^-)$ is integrable in ψ^- with integral bounded by 1. Thus, by the Fubini–Tonelli theorem, it is integrable in (ψ^+,ψ^-) and by the same theorem the function

$$\eta(\psi^+,\psi^-) = \int du\, \eta_u(\psi^+,\psi^-)$$

is integrable. It is the density of our measure η since the η measure of the set $\psi^+ = 0$ vanishes. The claim is proved. □

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