Analyticity of the Sinai-Ruelle-Bowen measure for a class of simple Anosov flows

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We consider perturbations of the Hamiltonian flow associated with the geodesic flow on a surface with constant negative curvature. We prove that, under a small perturbation, not necessarily of Hamiltonian character, the Sinai-Ruelle-Bowen measure associated with the flow exists and is analytic in the strength of the perturbation. An explicit example of "thermostated" dissipative dynamics is considered. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2747612]

I. INTRODUCTION

In recent time, much effort has been devoted to the analysis of hyperbolic systems, in part due to the *chaotic hypothesis*, introduced ten years ago in Ref. 14, which states that a many-particle system in a nonequilibrium stationary state behaves as a uniformly hyperbolic dynamical system (Anosov or more generally Axiom A system), at least for the purpose of evaluating macroscopic observables. This hypothesis can be seen as a generalization of the ergodic hypothesis to nonequilibrium systems, at least for systems in a stationary state. Although it is very hard to prove uniform hyperbolicity for realistic model systems, ideas connected with the chaotic hypothesis have played an important role in analyzing the results of numerical or real experiments.

Several results have been obtained in this direction, among which is the Gallavotti-Cohen *fluctuation theorem* (FT), a result concerning the large deviation functional of the phase space contraction rate (often identified with the *entropy production rate*), that extend the fluctuation-dissipation relation to systems in a nonequilibrium stationary state. The FT was proven rigorously in Ref. 12 for Anosov diffeomorphisms and then in Ref. 16 for Anosov flows. Furthermore several numerical tests have been conducted, using mathematical models of dissipative reversible systems and the chaotic hypothesis.

Most of the results quoted above are based on the existence of the Sinai-Ruelle-Bowen (SRB) measure. This existence was proven for a wide class of hyperbolic systems.[8,25] Unfortunately explicit expressions for the SRB measure are quite difficult to obtain and can be worked out only in particular cases, e.g., Anosov coupled lattice map,[3] while most of the models used in the simulations are based on continuous time dynamics (hyperbolic flows). We observe that, in order to obtain models for nonequilibrium systems, one cannot consider the simplest examples of Anosov systems that, being volume preserving, are not dissipative.

In this paper we explicitly construct the SRB measure for a family of Anosov flows that includes dissipative cases. The flows considered are perturbations of the geodesic flow on a surface of constant negative curvature. Such a flow can be seen as a Hamiltonian flow restricted to

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[a)]Electronic mail: amaricci@lps.u-psud.fr the surface of unit energy. We will mainly consider perturbation arising by adding a force to the Hamiltonian equations of motion. If the chosen force is conservative (i.e., coming from a potential), the system remains Hamiltonian and volume preserving so that the stationary measure is not singular with respect to the volume measure. Otherwise, if the perturbation is nonconservative, the system is expected to have a SRB measure singular with respect to the volume measure (dissipativity). Many of the models used in numerical works fall under this last category.

The geodesic motion on a surface with constant negative curvature is the simplest example of continuous time Anosov system. The *structural stability* of these systems, namely, the existence of the conjugation between two close flows, was first proven in Ref. 1. Later on, in Appendix A of Ref. 19 and in Refs. 18 and 10, very general results on the regularity of the pressure, hence also of the *topological entropy* and of the *equilibrium states*, were proven essentially using the contracting mapping theorem or implicit function theorem, a point of view introduced by Moser[21] and Mather.[20]

The above papers do not discuss the regularity of the *SRB state* in the case of an *analytic perturbation*. Although we think the above techniques might be effective also in such a situation, we present here a direct proof in the spirit of Refs. 5, 3, and 13. It is quite natural to directly construct the relevant dynamical quantities, such as the *conjugation function*, the *contraction rate of the unstable space*, and the mean values of continuous observables with respect to a *Gibbs state*, as the absolutely convergent perturbative expansion around the unperturbed system.

It would be very interesting to study a lattice of coupled Anosov flows as it was done for Anosov diffeomorphisms (see Ref. 3). In this case, coupling two flows already results in a very difficult problem. To obtain such a coupling is enough to consider the Hamiltonian flow generated by the Hamiltonian $H_\varepsilon(g_1,g_2)=H_0(g_1)+H_0(g_2)+\varepsilon V(g_1,g_2)$ for a suitable potential V analytic and Γ periodic in g_1 and g_2. The main difficulty here is that, for $\varepsilon=0$, one has that $H_0(g_i)$, $i=1,2$, are two independent conserved quantities, while for $\varepsilon\neq 0$ they are not conserved any more. This implies that the coupled system cannot be uniformly hyperbolic and most of the techniques used in this paper do not apply directly. Several works have addressed the problem of the SRB measure for nonuniformly hyperbolic systems, see, e.g., Ref. 17. We hope to come back on this problem in the future.

The paper is organized as follows. In Sec. II we introduce the systems we consider and state the main results of the paper. Sections. III–V contain the proof of these results. Finally the Appendix contains some technical computations.

II. MODEL AND MAIN RESULTS

A. The geodesic flow

The complex upper half plane $\mathbb{C}_+ \overset{\text{def}}{=} \{z\in\mathbb{C}:\mathrm{Im}(z)>0\}$, endowed with the metric

$$g = y^{-2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

is called the *Lobachevskii plane*. The isometries of this plane are given by the real, 2×2 matrices h with $\det h>0$, where, if $z\in\mathbb{C}_+$, the action of h on z is

$$zh \overset{\text{def}}{=} \frac{h_{11}z+h_{21}}{h_{12}z+h_{22}} \in \mathbb{C}_+ \quad \text{for } h \overset{\text{def}}{=} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

Observe that h and $h'=\lambda h$, for $\lambda\neq 0$, define the same transformation so that the isometries are naturally represented by the elements of PSL(2, $\mathbb{R}$).

A compact surface can be constructed from the Lobachevskii plane in the same way as the torus can be obtained from the plane $\mathbb{R}^2$. Given a Fuchsian subgroup $\Gamma\subset \mathrm{PSL}(2,\mathbb{R})$ (see Ref. 23 or 2 for the definition), we can consider the equivalence relation generated by its action on $\mathbb{C}_+$,

$$z \sim z' \Leftrightarrow \exists\, \gamma \in \Gamma, \quad | z = z'\gamma .$$

The quotient set, indicated with $\Sigma = \mathbb{C}_+/\Gamma$, is the most general compact analytic surface with constant negative curvature.

We will consider as *unperturbed dynamical system* the flow generated by

$$H_0(x,y,p_x,p_y) \stackrel{\text{def}}{=} \frac{y^2}{2}(p_x^2 + p_y^2) \tag{1}$$

on the cotangent bundle $\mathcal{M} \stackrel{\text{def}}{=} T^*\Sigma$. For any given energy $\mathcal{E} > 0$, the surface

$$\mathcal{M}_{\mathcal{E}} \stackrel{\text{def}}{=} \{(x,y,p_x,p_y) \in \mathcal{M}: H_0(x,y,p_x,p_y) \equiv \mathcal{E}\}$$

is a compact, invariant manifold. The geodesic flow on the surface Σ can be identified in a natural way with the flow generated by Eq. (1) restricted to $\mathcal{M}_1$, see Ref. 15.

To add a conservative force to such a system, we consider an analytic Γ-periodic function $\{V(z), z \in \mathbb{C}_+\}$ and the new Hamiltonian

$$H_\varepsilon = H_0 + \varepsilon V, \tag{2}$$

which generates the equations of motion

$$\begin{aligned} \dot{x} &= y^2 p_x, \quad \dot{p}_x = -\varepsilon(\partial V/\partial x), \\ \dot{y} &= y^2 p_y, \quad \dot{p}_y = -y(p_x^2 + p_y^2) - \varepsilon(\partial V/\partial y). \end{aligned} \tag{3}$$

We can then add a nonconservative force to our system. To obtain well defined equations of motion it has to be *covariant* with respect to the transformations in Γ. To obtain this we can consider the *automorphic* function of order 1, ϕ, defined by

$$\phi(z\gamma) = \phi(z) j^2(z,\gamma), \qquad \forall\, \gamma \in \Gamma,$$

where $j(z,h) \stackrel{\text{def}}{=} h_{12}z + h_{22}$, see Ref. 11. Setting

$$E_x = \frac{\overline{\phi(z)} + \phi(z)}{2}, \quad E_y = \frac{\overline{\phi(z)} - \phi(z)}{2i},$$

where $\bar{\phi}$ is the complex conjugate of ϕ, we obtain a force field which is locally conservative but is not the differential of a function. We can still define the *potential difference* between two points, $z, z_0 \in \Sigma$,

$$U(z) - U(z_0) \stackrel{\text{def}}{=} -\frac{1}{2}\int_{z_0}^{z} \mathrm{d}w\, \phi(w) - \frac{1}{2}\int_{z_0}^{z} \mathrm{d}\bar{w}\, \overline{\phi(w)}$$

as a multivalued function.

The energy H_ε computed along a motion which contains the force (E_x, E_y) tends asymptotically to increase. In order to keep it constant, we introduce a Gaussian thermostat, namely, a momentum-dependent friction of the form $\alpha(p) = p \cdot E/p^2$. Finally, the equations of motion for the *perturbed flow* on $\mathcal{M}^{\varepsilon}_{\mathcal{E}} \stackrel{\text{def}}{=} \{(x,y,p_x,p_y) \in \mathcal{M}: H_\varepsilon(x,y,p_x,p_y) \equiv \mathcal{E}\}$ are

$$\dot{x} = y^2 p_x, \quad \dot{p}_x = -\varepsilon(\partial V/\partial x) + \varepsilon'[E_x - \alpha(p)p_x],$$
$$\dot{y} = y^2 p_y, \quad \dot{p}_y = -y(p_x^2 + p_y^2) - \varepsilon(\partial V/\partial y) + \varepsilon'[E_y - \alpha(p)p_y], \tag{4}$$

where ε' is the strength of the nonconservative field. Since only notational complications would arise from considering $\varepsilon \neq \varepsilon'$, in the following, we will restrict ourselves to the $\varepsilon = \varepsilon'$ case. Under the dynamics in Eq. (4) H_ε is an integral of the motion.

B. Canonical coordinates

A simpler representation of the unperturbed dynamics was introduced in Ref. 9. We consider the *canonical transformation* from $\mathcal{M}\backslash\{H_0=0\}$ to $\mathcal{G} \stackrel{\text{def}}{=} \mathrm{GL}(2,\mathbb{R})/\Gamma$,

$$(p_x, p_y, x, y) \leftrightarrow \begin{pmatrix} p_1 & q_2 \\ -p_2 & q_1 \end{pmatrix} \stackrel{\text{def}}{=} g,$$

defined by

$$p_x + ip_y = (i/2)\det^2(g)\overline{j(i,g^{-1})}^2 = (i/2)(p_1 + iq_2)^2,$$
$$x + iy = ig^{-1} = (p_2 + iq_1)/(p_1 - iq_2). \tag{5}$$

This transforms the equations of motion in Eq. (3) into those generated by the new Hamiltonian (with slight abuse of notation, we still call H_ε and V the Hamiltonian and the potential as functions of the matrix g)

$$H_\varepsilon(g) \stackrel{\text{def}}{=} \frac{\det^2(g)}{8} + \varepsilon V(g). \tag{6}$$

Clearly H_ε is an analytic function of g. Introducing the matrices

$$\sigma^0 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^3 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^+ \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{7}$$

the Hamilton equation derived from Eq. (6) reads

$$\dot{g} = -\frac{\det(g)}{4} g\sigma^3 + \varepsilon\sigma^x \frac{\partial V}{\partial g}(g)\sigma^y \tag{8}$$

for $\sigma^x \stackrel{\text{def}}{=} (\sigma^+ + \sigma^-)$ and $\sigma^y \stackrel{\text{def}}{=} (\sigma^+ - \sigma^-)$. The nonconservative equations of motion in Eq. 4 reads

$$\dot{g} = -\frac{\det(g)}{4} g\sigma^3 + \varepsilon\sigma^x \frac{\partial V}{\partial g}(g)\sigma^y - \varepsilon c(g) g\sigma^y \stackrel{\text{def}}{=} \frac{\det(g)}{4}[-g\sigma^3 + \varepsilon\mathcal{F}(g)], \tag{9}$$

where the function $c(g)$ is

$$c(g) = \frac{1}{2\det^2(g)}\left[\frac{\phi(ig^{-1})}{j^2(i,g^{-1})} + \frac{\overline{\phi(ig^{-1})}}{j^2(-i,g^{-1})}\right].$$

This is an explicit example of a nonconservative system. Moreover it is possible to prove that the systems in Eq. (9) generically have a positive average space contraction rate, see Ref. 4.

Remark: Our techniques can be extended to a more general case. Given a Hamiltonian

$$H_\varepsilon(g) \overset{\text{def}}{=} H_0(g) + \varepsilon V(g), \tag{10}$$

like in Eq. (6) we can consider any analytic vector field $\mathcal{V}_\varepsilon$ on $\mathcal{M}$, ε close to the Hamiltonian vector field generated by H_0 and tangent to the level surfaces of H_ε. Clearly the flow generated by such a vector field preserves H_ε and the following results hold in this more general situation.

C. The conjugation

Let $\Phi_t:\mathcal{G}_\mathcal{E}\to\mathcal{G}_\mathcal{E}$ and $\Phi_t^\varepsilon:\mathcal{G}_\mathcal{E}^\varepsilon\to\mathcal{G}_\mathcal{E}^\varepsilon$ be the flows generated by the Hamiltonian H_0 and by the dissipative system in Eq. (9), respectively. As the first step we want to prove that these two flows can be conjugated by a change of coordinate. In contrast to the case of Anosov diffeomorphisms, this is not enough to map Φ_t into Φ_t^ε, but a local rescaling of time is also required. The details are given in the following theorem. To state it we need some notations:

$$\mathcal{G}_\mathcal{E} = \{g \in \mathcal{G} | H_0(g) = \mathcal{E}\}, \quad \mathcal{G}_{>\mathcal{E}} = \{g \in \mathcal{G} | H_0(g) > \mathcal{E}\},$$

$$\mathcal{G}_\mathcal{E}^\varepsilon = \{g \in \mathcal{G} | H_\varepsilon(g) = \mathcal{E}\}, \quad \mathcal{G}_{>\mathcal{E}}^\varepsilon = \{g \in \mathcal{G} | H_\varepsilon(g) > \mathcal{E}\}.$$

Theorem 1: Conjugation. *Given $\mathcal{E}>0$, there exists an $\bar{\varepsilon}>0$ such that, for any $\varepsilon:|\varepsilon|\le\bar{\varepsilon}$ there are functions $h_\varepsilon:\mathcal{G}_{>\mathcal{E}}\to\mathcal{G}_{>\mathcal{E}}^\varepsilon$, and $\tau_\varepsilon:\mathcal{G}_{>\mathcal{E}}\to\mathbb{R}$, Hölder continuous in g and analytic in ε, such that*

$$h_\varepsilon \circ \Phi_t = \Phi^\varepsilon_{T_t^\varepsilon} \circ h_\varepsilon \quad \text{for } T_t^\varepsilon \overset{\text{def}}{=} \int_0^t \mathrm{d}s(\tau_\varepsilon \circ \Phi_s). \tag{11}$$

Furthermore, $H_0 \equiv H_\varepsilon \circ h_\varepsilon$, so that $h_\varepsilon(\mathcal{G}_\mathcal{E}) = \mathcal{G}_\mathcal{E}^\varepsilon$. The proof, given in Sec. III, is based on the hyperbolicity of the unperturbed flow, which is discussed in the next section.

The function h_ε is the *space conjugation*, while τ_ε is the *time conjugation*. Even if h_ε conjugate the flow from $\mathcal{G}_\mathcal{E}$ to $\mathcal{G}_\mathcal{E}^\varepsilon$, the existence of a conjugation from the whole $\mathcal{G}$ to itself cannot be uniform in ε. Indeed, for fixed $\mathcal{E}$, if $\mathcal{E}<\varepsilon \sup_g V(g)$, the topology of $\mathcal{G}_\mathcal{E}^\varepsilon$ is different from that of $\mathcal{G}_\mathcal{E}$, and no conjugation is possible.

D. Hyperbolicity

If the tangent space $T_g\mathcal{G}_\mathcal{E}^\varepsilon$ can be split into three continuous, Φ^ε-covariant, one-dimensional, linear subspaces:

$$T_g\mathcal{G}_\mathcal{E}^\varepsilon = E_g^+ \oplus E_g^- \oplus E_g^3, \tag{12}$$

where E_g^3 is parallel to the flow and there exists constants $c,\lambda>0$ such that

$$\begin{aligned} \|(T_g\Phi_t^\varepsilon)w\| &\le ce^{-\lambda t}\|w\| \quad \text{for } w \in E_g^-, \quad t \ge 0, \\ \|(T_g\Phi_t^\varepsilon)w\| &\le ce^{\lambda t}\|w\| \quad \text{for } w \in E_g^+, \quad t \le 0, \end{aligned} \tag{13}$$

then the flow Φ^ε is *hyperbolic* on $\mathcal{G}_\mathcal{E}^\varepsilon$. Moreover E_g^+, E_g^-, and E_g^3 are called the *unstable*, *stable* and *neutral* subspaces, respectively.

The unperturbed flow Φ is hyperbolic on $\mathcal{G}_\mathcal{E}$ for every $\mathcal{E}>0$. The solution of Eq. (8) is explicitly given by

$$\Phi_t(g) \overset{\text{def}}{=} ge^{-(\det(g)/4)t\sigma_3} \bmod \Gamma \tag{14}$$

and it is clear that E_g^α is generated by $g\sigma^\alpha$ for $\alpha=\pm,3$ and $\lambda=\sqrt{2\mathcal{E}}$.

The four curves

$$\Phi_\zeta^\alpha(g) \stackrel{\text{def}}{=} g e^{-\zeta\sigma^\alpha} \bmod \Gamma \quad \text{for } \alpha = 3,0,\pm \tag{15}$$

are the integral manifold of the vector fields $w^a(g) \stackrel{\text{def}}{=} -g\sigma^a$ for $a=0,\pm,3$. We remark that $\Phi_t \equiv \Phi^3_{t\,\det(g)/4}$ and that Φ^0 is orthogonal to $\mathcal{G}_{\mathcal{E}}$.

Calling $\lambda^\pm(g) = \pm\det(g)/2 = \pm\sqrt{2H_0(g)}$ and $\lambda^3 \equiv 0$ the *Lyapunov exponents* of Φ_t and using that the commutation relation among the matrices $\{\sigma^i\}_{i=0,\,3,\,\pm}$ are

$$[\sigma^3,\sigma^+] = 2\sigma^+, \quad [\sigma^3,\sigma^-] = -2\sigma^-, \quad [\sigma^+,\sigma^-] = \sigma^3, \tag{16}$$

we obtain that

$$\Phi_t \circ \Phi_\zeta^\alpha = \Phi^\alpha_{\zeta\,\exp\{t\lambda^\alpha(g)\}} \circ \Phi_t. \tag{17}$$

Theorem 2: Hyperbolicity. *For any energy $\mathcal{E}>0$, there exists $\bar{\varepsilon}>0$ such that, for any $\varepsilon:|\varepsilon| \leq \bar{\varepsilon}$ the flow Φ^ε on $\mathcal{G}^\varepsilon_{\mathcal{E}'}$ is hyperbolic for every $\mathcal{E}'>\mathcal{E}$. In particular, there exist vector fields $\{w_\varepsilon^\alpha\}_{\alpha=0,\,\pm}$ and functions $\{\lambda_\varepsilon^\alpha\}_{\alpha=0,\,\pm}$ on $\mathcal{G}^\varepsilon_{>\mathcal{E}}$ such that*

$$T\Phi_t^\varepsilon w_\varepsilon^\alpha = \exp\left\{\int_0^t \mathrm{d}s(\lambda_\varepsilon^\alpha \circ \Phi_s^\varepsilon)\right\}(w_\varepsilon^\alpha \circ \Phi_t^\varepsilon) \quad \text{for } \alpha = 0,\pm. \tag{18}$$

Furthermore, $\{w_\varepsilon^\alpha \circ h_\varepsilon\}_{\alpha=0,\,\pm}$ and $\{\lambda_\varepsilon^\alpha \circ h_\varepsilon\}_{\alpha=0,\,\pm}$, are analytic in ε and Hölder continuous in g.

Notwithstanding that we called the conjugation a change of variables, since it is not differentiable but only Hölder continuous, this theorem is *not* a direct consequence of Theorem 1. The fact that $\{\lambda_\varepsilon^\alpha \circ h_\varepsilon\}_{\alpha=0,\,\pm}$, rather than $\{\lambda_\varepsilon^\alpha\}_{\alpha=0,\,\pm}$, are analytic in ε will be important for the construction of the SRB measure.

E. Sinai-Ruelle-Bowen distribution

For any energy $\mathcal{E}$ we can define the SRB measure on $\mathcal{G}^\varepsilon_{\mathcal{E}}$:

$$\mu^\varepsilon_{\mathcal{E}}(\mathcal{O}) \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t(\mathcal{O}\circ\Phi_t^\varepsilon)(g), \tag{19}$$

provided that such a limit exists and is constant Lebesgue almost everywhere in g for every continuous function $\mathcal{O}$. Such a measure exists, unique and ergodic, if the dynamical system is Anosov and topologically mixing, i.e., it is hyperbolic in the whole $\mathcal{G}^\varepsilon_{\mathcal{E}}$ and the stable and the unstable manifold are dense $\mathcal{G}_{\mathcal{E}}$.

The flow Φ is Anosov since it is also Hamiltonian; it is easy to prove that its SRB measure is the Lebesgue measure. Regarding Φ^ε, uniform hyperbolicity was established in Theorem 2, while topological mixing is a direct consequence of the existence of the conjugation.

Theorem 3: Analyticity of the SRB measure. *Given $\mathcal{E}>0$, there exists $\bar{\varepsilon}>0$, such that, for any $\varepsilon:|\varepsilon|<\bar{\varepsilon}$ the SRB measure $\mu^\varepsilon_{\mathcal{E}'}$ is analytic in ε for every $\mathcal{E}'>\mathcal{E}$, i.e., for any analytic $\mathcal{O}:\mathcal{G}\to\mathbb{R}$, the mean value $\mu^\varepsilon_{\mathcal{E}}(\mathcal{O})$ is analytic in ε.*

This is our main result. The proof will consist in an explicit construction of the SRB measure.

To summarize, for any energy $\mathcal{E}>0$ and ε small enough, we have constructed an hyperbolic structure and the corresponding SRB measure on each one of the leaves $\{\mathcal{G}^\varepsilon_{\mathcal{E}'}\}_{\mathcal{E}'\geq\mathcal{E}}$ in $\mathcal{G}^\varepsilon_{\geq\mathcal{E}}$. The set $\{\mu^\varepsilon_{\mathcal{E}'}\}_{\mathcal{E}'>\mathcal{E}}$ is an invariant measure on $\mathcal{G}^\varepsilon_{>\mathcal{E}}$.

III. PROOF OF THEOREM 1

A. Directional derivatives

For any smooth f on $\mathcal{G}$ we define the *directional derivative* along the curves $\{\Phi^\alpha\}_{\alpha=0,\ \pm,\ 3}$, as

$$(\mathcal{L}_\alpha f)(g) \stackrel{\text{def}}{=} \left. \frac{\mathrm{d}(f \circ \Phi^\alpha_\zeta)}{\mathrm{d}\zeta} \right|_{\zeta=0} (g). \tag{20}$$

These derivatives satisfy the relation $(\mathcal{L}_\alpha w^3) - (\mathcal{L}_3 w^\alpha) = \lambda^\alpha w^\alpha$. Since the stable, unstable, and neutral directions are tangent to $\mathcal{G}_\mathcal{E}$, whereas w^0 is transversal to it, we have

$$\begin{aligned} &(\mathcal{L}_\alpha H_0)(g) \equiv 0 \quad \text{for } \alpha = 3, \pm, \quad g \in \mathcal{G}_\mathcal{E}, \\ &(\mathcal{L}_0 H_0)(g) \neq 0 \quad \text{for } g \in \mathcal{G}_\mathcal{E}. \end{aligned} \tag{21}$$

Given $\gamma < 1$ and a function f on $\mathcal{G}$, we also define the *directional Hölder derivative* along $\{\Phi^\alpha\}_{\alpha=0,\pm,3}$ as

$$(\mathcal{L}^\gamma_\alpha f)(g) \stackrel{\text{def}}{=} \sup_{\zeta: 0 < |\zeta| \leq 1} \frac{|(f \circ \Phi^\alpha_\zeta)(g) - f(g)|}{|\zeta|^\gamma}, \tag{22}$$

if the supremum is finite.

B. Construction of the conjugation

In order to find a solution of Eq. (11), let us differentiate it with respect to t for $t=0$:

$$(\mathcal{L}_3 h_\varepsilon)(g) = \frac{(\det \circ h_\varepsilon)(g)}{\det(g)} \tau_\varepsilon(g)[w^3 \circ h_\varepsilon + \varepsilon \mathcal{F} \circ h_\varepsilon](g). \tag{23}$$

We will look for a solution h_ε and τ_ε of the form

$$\begin{aligned} h_\varepsilon(g) &= g + \sum_{\alpha=0,\ \pm,\ 3} \delta h^\alpha_\varepsilon(g) w^\alpha(g) = \sum_{\alpha=0,\ \pm,\ 3} [\delta_{0,\alpha} + \delta h^\alpha_\varepsilon(g)] w^\alpha(g), \\ \tau_\varepsilon &= 1 + \delta\tau_\varepsilon, \end{aligned} \tag{24}$$

where $\delta_{\alpha,\beta}$ is the Kronecker symbol. Projecting along the directions $\{w^\alpha(g)\}_{\alpha=0,\ \pm,\ 3}$ and using the identity following Eq. (1) yields (see the Appendix for the details):

$$(\mathcal{L}_3 \delta h^\alpha_\varepsilon)(g) - \lambda^\alpha \delta h^\alpha_\varepsilon(g) = \varepsilon \mathcal{F}^\alpha(g) + \mathcal{R}^\alpha_\varepsilon(\delta h^0_\varepsilon, \delta h^3_\varepsilon, \delta h^+_\varepsilon, \delta h^-_\varepsilon, \delta\tau_\varepsilon) + \delta_{\alpha,3}(\delta\tau_\varepsilon(g) - 2\,\delta h^0_\varepsilon(g)). \tag{25}$$

In the right-hand side of Eq. (25), $\{\mathcal{F}^\alpha : \mathcal{G} \to \mathbb{R}\}_{\alpha=0,\ \pm,\ 3}$ are analytic functions of g, depending neither on δh_e nor on $\delta\tau_\varepsilon$, while $\{\mathcal{R}^\alpha_\varepsilon : \mathbb{R}^5 \to \mathbb{R}\}_{\alpha=0,\ \pm,\ 3}$ are analytic functions of the form

$$\mathcal{R}^\alpha_\varepsilon(f_1, f_2, f_3, f_4, f_5) = \varepsilon \sum_{i=1}^{5} C^\mathcal{R}_{\alpha,i} f_i + O(f^2) \tag{26}$$

for suitable constants $\{C^\mathcal{R}_{\alpha,i}\}^{i=1,\ldots,5}_{\alpha=0,\ \pm,\ 3}$; the remainder $O(f^2)$ has Taylor expansion in ε starting from 0th order and Taylor expansion in the f starting from second order. The last term in Eq. (25) is $\delta_{\alpha,3}(f_5 - 2f_1)$; it is ε independent, but we singled it out because it is linear in f.

C. Implicit solution

We can implicitly solve Eq. (25). For every continuous $f : \mathcal{G} \to \mathbb{R}$, it is possible to invert the operators $\{\mathcal{L}_3 - \lambda^\beta\}_{\beta=\pm}$:

$$(\mathcal{L}_3-\lambda^{\beta})^{-1}f=\int_{\mathrm{sgn}(\lambda^{\beta})\infty}^{0}\mathrm{d}te^{(\mathcal{L}_3-\lambda^{\beta})t}f=\int_{\mathrm{sgn}(\lambda^{\beta})\infty}^{0}\mathrm{d}te^{-\lambda^{\beta}t}(f\circ\Phi_t),\quad \beta=\pm\,, \tag{27}$$

where the exponentiallly decaying factor guarantees convergence.

The implicit solutions for the stable and unstable components of the conjugation are then

$$\delta h_{\varepsilon}^{\beta}=\int_{\mathrm{sgn}(\beta)\infty}^{0}\mathrm{d}te^{-\lambda^{\beta}t}(\mathcal{R}_{\varepsilon}^{\beta}\circ\Phi_t)+\varepsilon\int_{\mathrm{sgn}(\beta)\infty}^{0}\mathrm{d}te^{-\lambda^{\beta}t}(\mathcal{F}_{\varepsilon}^{\beta}\circ\Phi_t),\quad \beta=\pm\,, \tag{28}$$

for $\mathcal{R}_{\varepsilon}^{\beta}\circ\Phi_t\overset{\mathrm{def}}{=}\mathcal{R}_{\varepsilon}^{\alpha}(\{\delta h_{\varepsilon}^{\alpha}\circ\Phi_t\}_{\alpha=0,\ \pm,\ 3},\delta\tau_{\varepsilon}\circ\Phi_t)$. The equation for $\delta h_{\varepsilon}^{3}$ cannot be solved in the same way since $\lambda^3\equiv 0$. Nonetheless, we can choose τ_{ε} so that the right-hand side member of Eq. (25), for $\alpha=3$, is identically zero:

$$\delta\tau_{\varepsilon}=2\,\delta h_{\varepsilon}^{0}-\varepsilon\mathcal{F}^{3}-\mathcal{R}_{\varepsilon}^{3}(\delta h_{\varepsilon}^{0},0,\delta h_{\varepsilon}^{+},\delta h_{\varepsilon}^{-},\delta\tau_{\varepsilon}). \tag{29}$$

Since $\lambda^0\equiv 0$ also, a similar problem occurs for the equation corresponding to $\delta h_{\varepsilon}^{0}$; in this case, it is possible to obtain an equation for $\delta h_{\varepsilon}^{0}$ using that $H_{\varepsilon}\circ h_{\varepsilon}=H_0$. Considering the transversality condition in Eq. (21) and the implicit equations for the level surfaces, one can solve Eq. (25) in terms of $\delta h_{\varepsilon}^{0}$ only, obtaining

$$\delta h_{\varepsilon}^{0}=-\frac{1}{\mathcal{L}_0H_0}\left[H_0\circ h_{\varepsilon}-H_0-\sum_{\alpha}(\mathcal{L}_{\alpha}H_0)\cdot\delta h_{\varepsilon}^{\alpha}+\varepsilon V\circ h_{\varepsilon}\right]\overset{\mathrm{def}}{=}-\varepsilon\frac{V}{\mathcal{L}_0H_0}-\mathcal{O}(\delta h_{\varepsilon}^{0},\delta h_{\varepsilon}^{3},\delta h_{\varepsilon}^{+},\delta h_{\varepsilon}^{-},\delta\tau_{\varepsilon}), \tag{30}$$

where $\mathcal{O}$ can be written as in Eq. (26) for certain other constants $\{C_{\alpha,i}^{\mathcal{O}}\}_{\alpha=0,\ \pm,\ 3}^{i=1,\ldots,5}$. The fact that w^0 is orthogonal to the level surfaces of H_0 [see Eq. (21)] guarantees that this expression is well defined for any $g\in\mathcal{G}_{\mathcal{E}}$ and ε small enough.

D. Existence of the conjugation

Observe that Eqs. (28)–(30) can be naturally seen as defining a function $f\overset{\mathrm{def}}{=}\{f^{\alpha}:\mathcal{G}_{\mathcal{E}}\to\mathbb{R}^4\}_{\alpha=0,\ \pm,\ 3}$ for $f^0=\delta h_{\varepsilon}^{0}$, $f^{\pm}=\delta h_{\varepsilon}^{\pm}$, and $f^3=\delta\tau_{\varepsilon}$. We will look for a solution of the above equations in the Banach space $\mathcal{B}$ defined by the norm $\|f\|_{\gamma}\overset{\mathrm{def}}{=}\max_{\alpha}\|f^{\alpha}\|_{\gamma}$, with

$$\|f^{\alpha}\|_{\gamma}\overset{\mathrm{def}}{=}\|f^{\alpha}\|+\sum_{\beta=\pm}\|\mathcal{L}_{\beta}^{\gamma}f^{\alpha}\|+\sum_{\beta=3,\ 0}\|\mathcal{L}_{\beta}f^{\alpha}\|,$$

where $\|u\|\overset{\mathrm{def}}{=}\sup_{g\in\mathcal{G}}|u(g)|$ for $u:\mathcal{G}\to\mathbb{R}$.

The equation for the conjugation is given in terms of the operator

$$(Lf)^{\alpha}\overset{\mathrm{def}}{=}\begin{cases}(\mathcal{L}_3-\lambda^{\alpha})f^{\alpha} & \text{if } \alpha=\pm\\ f^{\alpha} & \text{if } \alpha=0,3,\end{cases}$$

and the function $S_{\varepsilon}^{\alpha}(f)$ the components of which are defined as

$$\varepsilon\mathcal{F}^{\alpha}+\mathcal{R}_{\varepsilon}^{\alpha}(f^0,0,f^+,f^-,f^3)\quad(\alpha=\pm),$$

$$-\varepsilon(\mathcal{L}_0H_0)^{-1}\cdot V+\mathcal{O}(f^0,0,f^+,f^-,f^3)\quad(\alpha=0),$$

$$-\varepsilon[(\mathcal{L}_0H_0)^{-1}\cdot 2V+\mathcal{F}^3]-(2\mathcal{O}+\mathcal{R}_{\varepsilon}^{3})(f^0,0,f^+,f^-,f^3)\quad(\alpha=3).$$

Lemma 1: *There exists $\bar{\varepsilon}>0$ such that, for any $\varepsilon:|\varepsilon|\leq\bar{\varepsilon}$, the equation*

$$Lf = S_{\varepsilon}(f) \tag{31}$$

has a unique solution in the ball of $\mathcal{B}$ with radius $|\varepsilon|C$ for a suitable C. Such a solution is analytic in ε.

Proof: We first bound the norm of L^{-1}. From Eq. (17) it follows that

$$\sup_{|\zeta|>0}\frac{|f\circ\Phi_t\circ\Phi_{\zeta}^{\alpha}-f\circ\Phi_t|}{|\zeta|^{\gamma}} = e^{\gamma t\lambda^{\alpha}}\sup_{\zeta>0}\frac{|(f\circ\Phi^{\alpha}_{\zeta\exp\{t\lambda^{\alpha}\}}-f)\circ\Phi_t|}{|\zeta|^{\gamma}\exp\{\gamma t\lambda^{\alpha}\}}\leq e^{\gamma t\lambda^{\alpha}}(\|\mathcal{L}_{\alpha}^{\gamma}f\|+2\|f\|);$$

from this, it is easy to get the bound $\|L^{-1}\|_{\gamma}\leq 5/\lambda_{+}(1-\gamma)$.

We choose $C\geq\|L^{-1}\|\max\{1,4\|\mathcal{F}\|_{\gamma},4\|(\mathcal{L}_0H_0)^{-1}V\|_{\gamma}\}$. From Eq. (26), there exists a γ, ε-independent constant $C_0>1$ such that, for any $f,\tilde{f}$ in the ball $\mathcal{B}_{\varepsilon}\overset{\text{def}}{=}\{f\in\mathcal{B}:\|f\|_{\gamma}\leq|\varepsilon|C\}$,

$$\|\mathcal{O}(f)-\mathcal{O}(\tilde{f})\|_{\gamma},\quad \|\mathcal{R}^{\alpha}_{\varepsilon}(f)-\mathcal{R}^{\alpha}_{\varepsilon}(\tilde{f})\|_{\gamma}\leq|\varepsilon|C_0\|f-\tilde{f}\|_{\gamma}. \tag{32}$$

Indeed, it is possible to write $\mathcal{O}(f)-\mathcal{O}(\tilde{f})=\Sigma_{j=1}^{5}(f_j-\tilde{f}_j)\int_0^1\mathrm{d}t(\partial_j\mathcal{O})\circ(tf+(1-t)\tilde{f})$ and similarly for $\mathcal{R}^{\alpha}$; furthermore, the Hölder derivative of a product of functions is bounded by the product of the Hölder derivatives of each function. From Eq. (32) it follows

$$\|S^{\alpha}_{\varepsilon}(f)-S^{\alpha}_{\varepsilon}(\tilde{f})\|_{\gamma}\leq|\varepsilon|3C_0\|f-\tilde{f}\|_{\gamma}. \tag{33}$$

By the choice of C and using Eq. (33) for $\tilde{f}\equiv 0$, we have that, choosing $\bar{\varepsilon}=\lambda_{+}[(1-\gamma)/60C_0]$, $L^{-1}S_{\varepsilon}$ sends $\mathcal{B}_{\varepsilon}$ into itself. Moreover Eq. (33) implies that the application $L^{-1}S_{\varepsilon}$ is a contraction in $\mathcal{B}_{\varepsilon}$ since, by the previous choice, $\bar{\varepsilon}<\lambda_{+}[(1-\gamma)/20C_0]$. Since $\mathcal{F}$ and V are analytic, the solution of Eq. (31) is unique in $\mathcal{B}_{\varepsilon}$ and is the limit of a sequence of functions which are analytic in $\{\varepsilon\in\mathbb{C}:|\varepsilon|\leq\bar{\varepsilon}\}$. By Vitali theorem the solution is also analytic. ■

This Lemma concludes the proof of Theorem 1.

IV. PROOF OF THEOREM 2

A. Unstable direction

The second step toward the construction of an analytic SRB measure consists in constructing the perturbed unstable direction $w^{+}_{\varepsilon}(g)$ and the associated Lyapunov exponent $\lambda^{+}_{\varepsilon}(g)$. These quantities are both defined in Eq. (18).

As expected from the general theory,[1] the unstable direction of the perturbed system w^{+}_{ε} is *generically* not analytic in ε. To construct the SRB measure we need the unstable direction computed in the conjugated point h_{ε}, which we will see to be analytic in ε.

Calling $v^{+}_{\varepsilon}\overset{\text{def}}{=}w^{+}_{\varepsilon}\circ h_{\varepsilon}$ and $L_{\varepsilon}\overset{\text{def}}{=}\lambda^{+}_{\varepsilon}\circ h_{\varepsilon}\overset{\text{def}}{=}\lambda^{+}+\delta L_{\varepsilon}$, we will compute Eq. (18) for time t replaced by T^{ε}_{τ} and position $h_{\varepsilon}(g)$, rather than g. Using also Eq. (11), we obtain

$$(T_{h_{\varepsilon}}\Phi^{\varepsilon}_{T^{\varepsilon}_{t}})v^{+}_{\varepsilon}=e^{\int_0^t\mathrm{d}s(\tau_{\varepsilon}\circ\Phi_s)(L^{+}_{\varepsilon}\circ\Phi_s)}(v^{+}_{\varepsilon}\circ\Phi_t). \tag{34}$$

B. Construction of the unstable direction

Proceeding as in the previous section, taking the time derivative of both sides of Eq. (34) at $t=0$, we obtain

$$(T_{h_{\varepsilon}}\dot{\Phi}^{\varepsilon}_{0})v^{+}_{\varepsilon}-\frac{1}{\tau_{\varepsilon}}\frac{\det(g)}{4}(\mathcal{L}_3v^{+}_{\varepsilon})=L_{\varepsilon}\cdot v^{+}_{\varepsilon}. \tag{35}$$

We now write v_ε^+ as $v_\varepsilon^+ = w^+ + \Sigma_{a=0,3,-}\delta V_\varepsilon^a w^a$. Projecting along the direction w^+, calling $\mathcal{F}^{\cdot,+} \stackrel{\text{def}}{=} \mathcal{L}_+\mathcal{F}$, and defining $\mathcal{F}^\alpha$ such that $\mathcal{F} = \Sigma_{\alpha=0,3,\pm}\mathcal{F}^\alpha w^\alpha$ and $\mathcal{F}^{\alpha,+}$ such that $\mathcal{F}^{\cdot,+} = \Sigma_{\alpha=0,3,\pm}\mathcal{F}^{\alpha,+}w^\alpha$, after some lengthy but straightforward algebra, reported in the Appendix, we get

$$\delta L_\varepsilon = \frac{\det(g)}{4}[\varepsilon\mathcal{F}^{+,+}(g) - \delta\tau_\varepsilon(g) - \mathcal{P}_\varepsilon^+(\delta V_\varepsilon^0, \delta V_\varepsilon^3, \delta V_\varepsilon^-, \delta L_\varepsilon)], \tag{36}$$

while, projecting along the other directions, we get

$$[\mathcal{L}_3 - (\lambda^a - \lambda^+)]\delta V^a(g) = \varepsilon\mathcal{F}^{a,+}(g) - \delta_{a,3}2\delta V_\varepsilon^0(g) + \mathcal{P}_\varepsilon^a(\delta V_\varepsilon^0, \delta V_\varepsilon^3, \delta V_\varepsilon^-, \delta L_\varepsilon), \tag{37}$$

where $\{\mathcal{P}_\varepsilon^\alpha\}_{\alpha=0,\pm,3}$ can be written as in Eq. (26). In order to solve Eqs. (36) and (37), as for Eq. (25), we first need to replace the term $2\delta_{a,3}\delta V_\varepsilon^0$ in the right-hand side of Eq. (37), with the expression obtained by implicitly solving the equation for $\alpha=0$:

$$\delta V_\varepsilon^0(g) = \int_{-\infty}^0 \mathrm{d}s e^{s\lambda^+}[\varepsilon\mathcal{F}^{0,+} + \mathcal{P}_\varepsilon^0] \circ \Phi_s,$$

for $\mathcal{P}_\varepsilon^0 \circ \Phi_s \stackrel{\text{def}}{=} \mathcal{P}_\varepsilon^a(\{\delta V_\varepsilon^a \circ \Phi_s\}_{a=0,3,-}, \delta L_\varepsilon \circ \Phi_s)$. Substituting into Eq. (37), we get

$$[\mathcal{L}_3 - (\lambda^a - \lambda^+)]\delta V^a = \varepsilon\tilde{\mathcal{F}}^{a,+} + \tilde{\mathcal{P}}_\varepsilon^a(\delta V_\varepsilon^0, \delta V_\varepsilon^3, \delta V_\varepsilon^-, \delta L_\varepsilon) \tag{38}$$

for suitable $\{\tilde{\mathcal{F}}^{a,+}\}_{a=0,}$ which depend neither on $\{\delta V_\varepsilon^a\}_{a=0,-,0,3}$ nor on δL_ε, and is linear in ε and Hölder continuous in g. Moreover, $\{\tilde{\mathcal{P}}_\varepsilon^a\}_{a=0,-,3}$ are analytic in their arguments and can be written as in Eq. (26) for suitable constants $\{\tilde{C}_{j,a}\}_{a=0,-,3}^{j=1,\ldots,4}$.

C. Existence of the perturbed unstable direction

Calling $f^0 = \delta V_\varepsilon^0$, $f^3 = \delta V_\varepsilon^3$, $f^- = \delta V_\varepsilon^-$, and $f^+ = \delta L_\varepsilon^+$, we can look for a solution of Eqs. (36) and (37) in the Banach space $\mathcal{B}$ introduced in Sec. III D. Again we introduce the operator:

$$(Mf)^\alpha \stackrel{\text{def}}{=} \begin{cases} f^\alpha & \text{if } \alpha = + \\ \mathcal{L}_3 - (\lambda^a - \lambda^+) & \text{if } \alpha = -,0,3, \end{cases}$$

and the function

$$T_\varepsilon^\alpha(f) \stackrel{\text{def}}{=} \begin{cases} \varepsilon\dfrac{\det(g)}{4}\mathcal{F}^{+,+} - \delta\tau_\varepsilon\dfrac{\det(g)}{4} + \mathcal{P}_\varepsilon^+(f^0,f^3,f^-,f^+) & (\alpha = +) \\ \varepsilon\tilde{\mathcal{F}}^{a,+} + \tilde{\mathcal{P}}_\varepsilon^a(f^0,f^3,f^-,f^+) & (\alpha = -,0,3), \end{cases} \tag{39}$$

and we prove the following lemma.

Lemma 2: *There exists $\bar{\varepsilon} > 0$ such that, for any $\varepsilon : |\varepsilon| \leq \bar{\varepsilon}$, the equation*

$$Mf = T_\varepsilon(f) \tag{40}$$

has a unique solution in the ball of $\mathcal{B}$ of radius εC for a suitable C. Such a solution is analytic in ε.

Proof: It follows from arguments similar to those used in the proof of Lemma 1. ■

Clearly the perturbed stable direction and Lyapunov exponent could be constructed in the very same way.

V. PROOF OF THEOREM 3

A. Markov partition

It is worthwhile to remark that for topologically mixing Anosov flows, the foliations E^+ and E^- are not jointly integrable and therefore it is not possible to find a surface which contains a finite piece of the stable and unstable manifold of a given point (see Ref. 22). This is why the following construction of the Markov partition,[7,24] is slightly different from a naive generalization of the construction of a Markov partition for diffeomorphisms.

We first consider the unperturbed flow Φ. By fixing $\delta > 0$, we define the local weak-stable and weak-unstable manifolds passing through g as

$$W_\delta^{3,\pm}(g) \overset{\text{def}}{=} \{(\Phi_t \circ \Phi_\zeta^\pm)(g) : |\zeta|, |t| < \delta\},$$

which are clearly C^ω manifolds. Let D be any closed C^ω disk of dimension 2 in $\mathcal{G}_\mathcal{E}$, transverse in each point to the flow Φ. Given two close points on D, g, g' with $d(g,g') \leqslant \alpha_1$, for α_1 small enough,

$$\langle g, g' \rangle_D \overset{\text{def}}{=} W_\delta^{3,-}(g) \cap W_\delta^{3,+}(g') \cap D \tag{41}$$

consists of a single point. We will say that T is a rectangle on D if $\log g, g' RG_D \in T$ for any $g, g' \in T$.

The two manifolds $W_T^-(g) = \{\log g, g' RG_D : g' \in T\}$ and $W_T^+(g) = \{\log g', gRG_D : g' \in T\}$ are the projections of the stable and unstable manifolds through g on the rectangle T, which can be seen as

$$T \equiv \langle W_T^+(g), W_T^-(g) \rangle. \tag{42}$$

Given a family of closed rectangles $\{T_1, \ldots, T_N\}$ on disks $\{D_1, \ldots, D_N\}$ such that $T_i \subset \operatorname{int} D_i$ and $T_i = \overline{\operatorname{int} T_i}$, we will call it a *proper family of rectangles* if there exists $\alpha > 0$ such that $\mathcal{G}_\mathcal{E} = \cup_{j=1}^N \cup_{t \in [0,\alpha]} \Phi_{-t}(T_j)$; for any $i \neq j$, at least one of the sets $D_i \cap [\cup_{t \in [0,\alpha]} \Phi_t(D_j)]$ and $D_j \cap [\cup_{t \in [0,\alpha]} \Phi_t(D_j)]$ is empty.

Let $\Pi = \cup_{j=1}^N T_j$ and define the *ceiling function* $\vartheta : \Pi \to \mathbb{R}_+$ as the smallest strictly positive time required for $\Phi_t(g)$ to cross Π and the *Poincaré map* $\mathcal{H} : \Pi \to \Pi$, as $\mathcal{H}(g) = \Phi_{\vartheta(g)}(g)$.

Finally, the proper family of rectangles, $\{T_1, \ldots, T_N\}$, is called *Markov partition* if it satisfies the following condition: for any $g \in T_i$ such that $\mathcal{H}^{\pm 1}(g) \in T_j$ one has that $\mathcal{H}^{\pm 1}(W^{\mp}(g)) \subset T_j$. In particular, it is possible to show that the flow Φ admits a Markov partition of the rectangles $\{T_1, \ldots, T_N\}$ on disks $\{D_1, \ldots, D_N\}$.

B. Symbolic dynamics

Let A be the incidence matrix associated with $\mathcal{H}$, i.e.,

$$A_{i,j} = \begin{cases} 1 & \text{if } \operatorname{int} T_i \cap \mathcal{H}(\operatorname{int} T_j) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

By the results in Refs. 7 and 24, we may suppose that there exists an integer k such that the matrix A^k has only nonzero entries. We introduce the space of sequences

$$\Sigma_A \overset{\text{def}}{=} \{\underline{\sigma} \in \{1, \ldots, N\}^\mathbb{Z} : A_{\sigma_i, \sigma_{i+1}} = 1, i \in \mathbb{Z}\},$$

the *shift map*, $\rho : \Sigma_A \to \Sigma_A$, such that $(\rho\underline{\sigma})_j = \sigma_{j+1}$ and the *coding map*, $X : \Sigma_A \to \Pi$, such that $X(\underline{\sigma}) \overset{\text{def}}{=} \cap_{i=-\infty}^{+\infty} \overline{\mathcal{H}^{-i}(\operatorname{int} T_{\sigma_i})}$; we remark that $\mathcal{H} \circ X = X \circ \rho$. Let $\nu(\underline{\sigma}, \underline{\sigma})$ be $\max\{n \in \mathbb{N} \cup \{0\} : \sigma_i = \sigma_i' \, \forall i : |i|$

$\leqslant n\}$, if at least $\sigma_0=\sigma_0'$; otherwise $\nu(\underline{\sigma},\underline{\sigma})\stackrel{\text{def}}{=}-1$. Endowing the space Σ_A with the distance $|\underline{\sigma}-\underline{\sigma}|=e^{-\nu(\underline{\sigma},\underline{\sigma})}$, the map ρ is continuous, and X is Hölder continuous.

Finally, the coding is inherited by all $g\in\mathcal{G}_{\mathcal{E}}$. After calling

$$Y\stackrel{\text{def}}{=}\{(\underline{\sigma},t)\in\Sigma_A\times\mathbb{R}_+:0\leqslant t\leqslant(\vartheta\circ X)(\underline{\sigma})\}$$

and identifying $(\underline{\sigma},(\vartheta\circ X)(\underline{\sigma}))$ with $(\rho\underline{\sigma},0)$, let $q:Y\to\mathcal{G}_{\mathcal{E}}$ be the one-to-one map defined by $q(\underline{\sigma},t)=(\Phi_t\circ X)(\underline{\sigma})$; then

$$(\Phi_t\circ q)(\underline{\sigma},s)=q(\rho^k\underline{\sigma},t') \tag{43}$$

for the unique k such that $t'\stackrel{\text{def}}{=}t+s-\Sigma_{j=0}^{k-1}(\vartheta\circ X\circ\rho^j)(\underline{\sigma})$ satisfies $0\leqslant t'<(\vartheta\circ X\circ\rho^k)(g)$.

C. Sinai-Ruelle-Bowen measure

Given a Hölder continuous $f:\Sigma_a\to\mathbb{R}$, we can associate with it the *equilibrium state with potential f*, *i.e.*, a ρ invariant, Gibbs measure ν_f on Σ_A, defined by the formal Hamiltonian

$$H(\underline{\sigma})\stackrel{\text{def}}{=}\sum_{j=-\infty}^{+\infty}f(\rho^j\underline{\sigma}), \tag{44}$$

see Ref. 6 for proofs and details.

Now, let $\Lambda_t^+(g)$ be the Jacobian of the linear map $T\Phi_t:E_g^+\to E_{\Phi_t(g)}^+$ and let

$$\lambda^+(g)\stackrel{\text{def}}{=}-\left.\frac{\mathrm{d}\ln\Lambda_t^+(g)}{\mathrm{d}t}\right|_{t=0}$$

which exists and is analytic in g. Finally we define the potential $\hat{f}^+$ as

$$\hat{f}^+(g)\stackrel{\text{def}}{=}\int_0^{\vartheta(g)}\mathrm{d}s(\lambda^+\circ\Phi_s)(g).$$

Given a continuous function $\mathcal{O}$ on $\mathcal{G}_{\mathcal{E}}$, the SRB measure $\mu_{\mathcal{E}}$ for Φ is given by

$$\mu_{\mathcal{E}}(\mathcal{O})=\nu_{\hat{f}^+\circ X}(\hat{\mathcal{O}}\circ X),$$

where

$$\hat{\mathcal{O}}(g)\stackrel{\text{def}}{=}\int_0^{\vartheta(g)}\mathrm{d}s(\mathcal{O}\circ\Phi_s)(g),$$

see Ref. 8 (Theorem 5.1). Since Φ is a Hamiltonian flow, μ is the Lebesgue measure.

For the perturbed, non-Hamiltonian flow Φ_t^ε, the SRB measure is generally not absolutely continuous with respect to the Lebesgue measure. Contrary to the naive expectation, the rectangles $\widetilde{T}_j^\varepsilon\stackrel{\text{def}}{=}h_\varepsilon(T_j)$ do not yield a Markov partition, since they are not portions of smooth disks.

We first observe that the disk D_i, $i=1,\ldots,N$, can be seen as the intersection of a smooth disk $\mathcal{D}_i$ of dimension 3 in $\mathcal{G}$ with the energy surface $\mathcal{G}_{\mathcal{E}}$. In this way we can define the disks $D_i^\varepsilon=\mathcal{D}_i\cup\mathcal{G}_{\mathcal{E}}^\varepsilon$. Let now Δ_i be the open neighbor of $\mathcal{D}_i$ defined by $\Delta_i=\cup_{t:|t|<\delta}\Phi_t^\varepsilon(\mathcal{D}_i)$. On $\cup_{i=1}^N\Delta_i$ we can define, for ε small enough, the maps $s_\varepsilon(g)$ as the solution of $\Phi_{s_\varepsilon(g)}^\varepsilon(g)\in\mathcal{D}_i$ and $q_\varepsilon(g)=\Phi_{s_\varepsilon(g)}^\varepsilon(g)$.

We can define the map $p_\varepsilon:\cup_{i=1}^N D_i\to\cup_{i=1}^N D_i^\varepsilon$ as

$$p_\varepsilon(g) \stackrel{\text{def}}{=} q_\varepsilon \circ h_\varepsilon(g),$$

which is clearly analytic in ε and Hölder continuous in g. It is easy to see that the sets $T_i^\varepsilon = p_\varepsilon(T_i)$ form a Markov partition for Φ^ε on $\mathcal{G}_\mathcal{E}^\varepsilon$. We can define the perturbed ceiling function $\vartheta_\varepsilon : \Pi^\varepsilon \stackrel{\text{def}}{=} \cup_{j=1}^N T_j^\varepsilon \to \mathbb{R}_+$ and the perturbed Poincaré map $\mathcal{H}_\varepsilon : \Pi^\varepsilon \to \Pi^\varepsilon$ as $\mathcal{H}_\varepsilon(g) \stackrel{\text{def}}{=} \Phi^\varepsilon_{\vartheta_\varepsilon(g)}(g)$. Clearly p_ε conjugates $\mathcal{H}$ with $\mathcal{H}_\varepsilon$. Finally, the coding map for the perturbed flow Φ_ε, $X_\varepsilon : \Sigma_A \to \Pi^\varepsilon$ is given by $X_\varepsilon = p_\varepsilon \circ X$.

Given a Hölder continuous function $\mathcal{O}$, its average with respect to $\mu_\mathcal{E}^\varepsilon$ is given by

$$\mu_\mathcal{E}^\varepsilon(\mathcal{O}) = \nu_{\hat{f}_\varepsilon^+ \circ X_\varepsilon}(\hat{\mathcal{O}}_\varepsilon \circ X_\varepsilon), \tag{45}$$

where $\hat{f}_\varepsilon^+, \hat{\mathcal{O}}_\varepsilon : \Pi^\varepsilon \to \mathbb{R}$ are defined as before:

$$\hat{f}_\varepsilon^+(g) \stackrel{\text{def}}{=} \int_0^{\vartheta_\varepsilon(g)} \mathrm{d}s(\lambda_\varepsilon^+ \circ \Phi_s^\varepsilon)(g), \quad \hat{\mathcal{O}}_\varepsilon(g) \stackrel{\text{def}}{=} \int_0^{\vartheta_\varepsilon(g)} \mathrm{d}s(\mathcal{O} \circ \Phi_s^\varepsilon)(g).$$

We observe that $(\Phi_s^\varepsilon \circ p_\varepsilon)(g) = (\Phi^\varepsilon_{s+(s_\varepsilon \circ h_\varepsilon)(g)} \circ h_\varepsilon)(g)$. Calling $\tilde{\vartheta}_\varepsilon : \cup_{j=1}^N \tilde{T}_j^\varepsilon \to \mathbb{R}_+$ the ceiling function for the Hölder continuous manifold $\cup_{j=1}^N \tilde{T}_j^\varepsilon$, we also have $(\vartheta_\varepsilon \circ p_\varepsilon)(g) = (\tilde{\vartheta}_\varepsilon \circ h_\varepsilon)(g) + (s_\varepsilon \circ h_\varepsilon \circ \mathcal{H})(g) - (s_\varepsilon \circ h_\varepsilon)(g)$. Therefore,

$$\begin{aligned}(\hat{f}_\varepsilon^+ \circ p_\varepsilon)(g) &= \int_0^{(\vartheta_\varepsilon \circ p_\varepsilon)(g)} \mathrm{d}s(\lambda_\varepsilon^+ \circ \Phi_s^\varepsilon \circ p_\varepsilon)(g) = \int_{(s_\varepsilon \circ h_\varepsilon)(g)}^{(\tilde{\vartheta}_\varepsilon \circ h_\varepsilon)(g) + (s_\varepsilon \circ h_\varepsilon \circ \mathcal{H})(g)} \mathrm{d}s(\lambda_\varepsilon^+ \circ \Phi_s^\varepsilon \circ h_\varepsilon)(g) \\ &\stackrel{\text{def}}{=} \int_0^{(\tilde{\vartheta}_\varepsilon \circ h_\varepsilon)(g)} \mathrm{d}s(\lambda_\varepsilon^+ \circ \Phi_s^\varepsilon \circ h_\varepsilon)(g) + (\hat{F}_\varepsilon^+ \circ \mathcal{H})(g) - \hat{F}_\varepsilon^+(g)\end{aligned} \tag{46}$$

for a suitable, Hölder continuous function $\hat{F}_\varepsilon^+ : \Pi \to \mathbb{R}$. It is well known that, due to its *cocycle* structure, the term $(\hat{F}_\varepsilon^+ \circ \mathcal{H})(g) - \hat{F}_\varepsilon^+(g)$ in the last line of Eq. (46) can be neglected. In the remaining integral we perform the change of integration variable from s to $s' : s = T_{s'}^\varepsilon(g)$ and we use the identities $(\tilde{\vartheta}_\varepsilon \circ h_\varepsilon)(g) = T^\varepsilon_{\vartheta(g)}(g)$ and $(\Phi^\varepsilon_{T_s^\varepsilon(g)} \circ h_\varepsilon)(g) = (h_\varepsilon \circ \Phi_s)(g)$ to get

$$\int_0^{(\tilde{\vartheta}_\varepsilon \circ h_\varepsilon)(g)} \mathrm{d}s(\lambda_\varepsilon^+ \circ \Phi_s^\varepsilon \circ h_\varepsilon)(g) = \int_0^{\vartheta(g)} \mathrm{d}s[\tau_\varepsilon \cdot (\lambda_\varepsilon^+ \circ h_\varepsilon) \circ \Phi_s](g).$$

The last expression is clearly analytic in ε due to the analyticity of τ_ε and of $\lambda_\varepsilon^+ \circ h_\varepsilon \equiv L_\varepsilon^+$. Observe that this integral is the potential we would have obtained considering directly the set $\tilde{T}_i^\varepsilon$ as a Markov partition.

To conclude the proof it is enough to observe that $\hat{\mathcal{O}}_\varepsilon \circ X_\varepsilon$ is clearly analytic in ε since $\vartheta_\varepsilon \circ p_\varepsilon(g)$ is. This implies that $\nu_{\hat{f}_\varepsilon^+ \circ X_\varepsilon}(\hat{\mathcal{O}}_\varepsilon \circ X_\varepsilon)$ is the average, with respect to a Gibbs state defined by potentials analytically depending on ε, of a function analytically depending on ε.

The theorem follows from standard results on Gibbs states, see Ref. 13: Hölder-continuous potentials can be converted into a many-body, exponentially vanishing interaction among spins (ranging in $\{1, \ldots, N\}$) that are placed on the sites of the lattice $\mathbb{Z}$ and that are also subjected to a "hard core" interaction (the compatibility condition associated with the matrix A). For such a system, the analyticity of the Gibbs measure with respect to the interaction can be obtained by cluster expansion.

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APPENDIX: EXPLICIT COMPUTATIONS

1. Explanation of Equation (25)

Taking the time derivative in $t=0$, the left-hand side of the first equation in Eq. (11) gives

$$\frac{\det(g)}{4}\left[w^3(g)+\sum_{\alpha=0,\ \pm,\ 3}\delta h_\varepsilon^\alpha(g)(\mathcal{L}_3 w^\alpha)(g)+\sum_{\alpha=0,\ \pm,\ 3}(\mathcal{L}_3\delta h_\varepsilon^\alpha)(g)w^\alpha(g)\right].$$

Therefore Eq. (25) follows from the identity

$$(w^3\circ h_\varepsilon)(g)=w^3+\sum_{\alpha=0,\ \pm,\ 3}\delta h_\varepsilon^\alpha(g)(\mathcal{L}_\alpha w^3)(g)$$

and from Eq. (24), which gives

$$\frac{(\det\circ h_\varepsilon)(g)}{\det(g)}=1-2\delta h_\varepsilon^0(g)+(\delta h_\varepsilon^0)^2(g)-(\delta h_\varepsilon^3)^2(g)-\delta h_\varepsilon^+(g)\delta h_\varepsilon^-(g).$$

2. Explanation of Equations (36) and (37)

Using the decomposition for v_ε after Eq. (35), Eq. (35) reads

$$\begin{aligned}(\mathcal{L}_+\dot{\Phi}_0^\varepsilon)(g)&+\sum_{a=0,\ 3,\ -}(\mathcal{L}_a\dot{\Phi}_0^\varepsilon)(g)\delta V^a(g)-\frac{1}{\tau_\varepsilon(g)}\frac{\det(g)}{4}(\mathcal{L}_3 w^+)(g)-\frac{1}{\tau_\varepsilon(g)}\frac{\det(g)}{4}\\&\times\sum_{a=0,\ 3,\ -}\delta V^a(g)(\mathcal{L}_3 w^a)(g)-\frac{1}{\tau_\varepsilon(g)}\frac{\det(g)}{4}\sum_{a=0,\ 3,\ -}w^a(g)(\mathcal{L}_3\delta V^a)(g)\\&=L_\varepsilon(g)w^+(g)+L_\varepsilon(g)\sum_{a=0,\ 3,\ -}\delta V^a(g)w^a(g)+(T_{h_e(g)}\dot{\Phi}_0^\varepsilon-T_g\dot{\Phi}_0^\varepsilon)v_\varepsilon(g).\end{aligned}\tag{A1}$$

From Eq. (9) we get

$$\begin{aligned}\frac{\det(g)}{4}(\mathcal{L}_+w^3)&+\frac{\det(g)}{4}\sum_{a=0,\ 3,\ -}(\mathcal{L}_a w^3)\delta V^a+\frac{\det(g)}{2}w^3\delta V^0-\frac{1}{\tau_\varepsilon}\frac{\det(g)}{4}(\mathcal{L}_3 w^+)\\&-\frac{1}{\tau_\varepsilon}\frac{\det(g)}{4}\sum_{a=0,\ 3,\ -}\delta V^a(\mathcal{L}_3 w^a)-\frac{1}{\tau_\varepsilon}\frac{\det(g)}{4}\sum_{a=0,\ 3,\ -}w^a(\mathcal{L}_3\delta V^a)\\&=L_\varepsilon\cdot w^++L_\varepsilon\sum_{a=0,\ 3,\ -}\delta V^a w^a-\varepsilon\frac{\det(g)}{4}(\mathcal{L}_+\mathcal{F})-\varepsilon\frac{\det(g)}{4}\sum_{a=0,\ 3,\ -}(\mathcal{L}_a\mathcal{F})\delta V^a-\varepsilon\frac{\det(g)}{2}\mathcal{F}\delta V^0\\&+(T_{h_e(g)}\dot{\Phi}_0^\varepsilon-T_g\dot{\Phi}_0^\varepsilon)v_\varepsilon(g).\end{aligned}\tag{A2}$$

Using the identity following Eq. (20) and the decomposition $L_\varepsilon=\lambda^++\delta L_\varepsilon$, we obtain

$$\delta\tau_\varepsilon(\mathcal{L}_+ w^3) - \sum_{a=0,\ 3,\ -} (\mathcal{L}_3 \delta V^a - (\lambda^a - \lambda^+)\delta V^a) w^a + 2w^3 \delta V^0 = \frac{4}{\det} \delta L_\varepsilon \cdot w^+ - \varepsilon(\mathcal{L}_+ \mathcal{F}) + \mathcal{P}_\varepsilon(\delta V^0_\varepsilon, \delta V^3_\varepsilon, \delta V^-_\varepsilon, \delta L_\varepsilon). \tag{A3}$$

Projecting along the direction w^+, calling $\mathcal{F}^{,\alpha} \stackrel{\text{def}}{=} \mathcal{L}_\alpha \mathcal{F}$, defining $\mathcal{F}^\alpha, \mathcal{P}^\alpha$ such that $\mathcal{F} = \Sigma_{\alpha=0,3,\pm} \mathcal{F}^\alpha w^\alpha$ and similarly for $\mathcal{P}^\alpha$, and finally defining $\mathcal{F}^{\alpha,\beta}$ such that $\mathcal{F}^{,\beta} = \Sigma_{\alpha=0,3,\pm} \mathcal{F}^{\alpha,\beta} w^\alpha$, we get Eqs. (36) and (37).

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