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Nonequilibrium stationary state of a current-carrying thermostated system

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Abstract – We find an explicit expression for the long time evolution and stationary speed distribution of N point particles in 2D moving under the action of a weak external field $\mathbf{E}$, and undergoing elastic collisions with either a fixed periodic array of convex scatterers, or with virtual random scatterers. The total kinetic energy of the N-particles is kept fixed by a Gaussian thermostat which induces an interaction between the particles. We show analytically and numerically that for weak fields this distribution is universal, *i.e.*, independent of the position or shape of the obstacles, as far as they form a dispersing billiard with finite horizon, or the nature of the stochastic scattering. Our results are nonperturbative. They exploit the existence of two time scales; the velocity directions become uniformized in times of order unity while the speeds change only on a time scale of $O(|\mathbf{E}|^{-2})$.

Introduction. – Our understanding of nonequilibrium stationary states (NESS) of multi-particle systems, arguably the simplest nonequilibrium systems, is very incomplete at present. In particular, there are no cases (we know of) where one has an explicit expression for the NESS of an interacting system of particles with positions and velocities [1,2]. In this paper we derive an analytic expression for a nontrivial NESS having a certain universality. While our proof requires some technical assumptions the arguments are physically clear and convincing [3]. The results are furthermore checked by very extensive computer simulations.

The model we consider is a variation of the Drude-Lorentz model of electrical conduction in two dimensions [4]. The system consists of N-particles (electrons) moving under the action of a constant external field $\mathbf{E}$ among a periodic array of fixed convex scatterers (Sinai billiard) with which they collide elastically. In order to produce a stationary current-carrying state it is necessary to have a mechanism which will absorb the heat produced by the field $\mathbf{E}$. This is modeled here by a Gaussian thermostat which keeps the total kinetic energy of the system constant [5–9]. This leads to a time evolution and a NESS whose properties we study here for weak fields. We find explicit expressions for both the speed distribution and current when $\mathbf{E}$ is small. The current is given by a Green-Kubo formula [10]. As in the Drude-Lorentz model we neglect direct interactions between the particles. We expect that the effect of such weak interaction will only slightly modify the property of our system for small but nonvanishing fields [3].

Dynamics. – The equations of motion for the system on the unit two-dimensional torus, which corresponds to an infinite system with periodic scatterers and periodic initial conditions are, taking the mass of each particle to be one, [9,11,12]

$$\begin{cases} \dot{\mathbf{q}}_i = \mathbf{v}_i, & i = 1, \dots, N, \\ \dot{\mathbf{v}}_i = \mathbf{F}_i = \mathbf{E} - \dfrac{\mathbf{E}\cdot\mathbf{J}}{U}\mathbf{v}_i + \mathcal{F}_i, & \end{cases} \tag{1}$$

where

$$\mathbf{J} = \sum_{i=1}^{N} \mathbf{v}_i, \qquad U = \sum_{i=1}^{N} |\mathbf{v}_i|^2 \tag{2}$$

and $\mathcal{F}_i$ is the "force" exerted on the i-th particle by collisions with the fixed scatterers. These collisions only change the direction but not the speed of the particle. It is easy to see that due to the Gaussian thermostat,

$$\begin{aligned}\frac{\partial W(\mathbf{Q},\mathbf{V},t)}{\partial t} &= -\sum_{i=1}^{N}\mathbf{v}_i\frac{\partial W(\mathbf{Q},\mathbf{V},t)}{\partial \mathbf{q}_i} - \sum_{i=1}^{N}\frac{\partial}{\partial \mathbf{v}_i}\left[(\mathbf{E}-(\mathbf{E}\cdot\mathbf{j})\mathbf{v}_i)W(\mathbf{Q},\mathbf{V},t)\right]\\ &\quad+\sum_{i=1}^{N}\frac{1}{2}\int_{(\mathbf{v}'_i\cdot\hat{\mathbf{n}})<0}\lambda(\mathbf{q}_i)(\mathbf{v}'_i\cdot\hat{\mathbf{n}})\left[W(\mathbf{Q},\mathbf{V}'_i,t;\mathbf{E})-W(\mathbf{Q},\mathbf{V},t;\mathbf{E})\right]\mathrm{d}\hat{\mathbf{n}}\\ &= \mathcal{A}W+E\,\mathcal{B}W+\mathcal{C}W, \end{aligned} \tag{4}$$

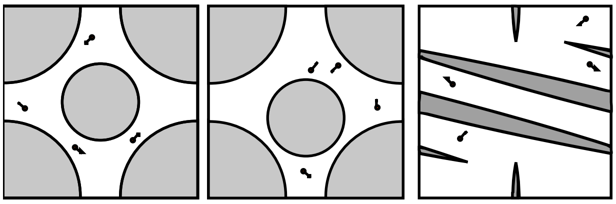

Fig. 1: (Colour on-line) Tables used for the simulations.

represented by the $(\mathbf{E}\cdot\mathbf{J})\mathbf{v}_i/U$ term, the total kinetic energy of the system is constant, *i.e.*, $\frac{\mathrm{d}}{\mathrm{d}t}U=0$.

This system was first introduced, for the case $N=1$, by Moran and Hoover [5] where it was found numerically that the NESS had a fractal structure. This was shown rigorously in [13], where it was proven that this system has a unique singular SRB (Sinai-Ruelle-Bowen) measure for small $\mathbf{E}$ which satisfies Ohm's law. The $N=1$ system was further investigated both numerically and analytically in [11,14].

In the prsent work we investigate the behavior of the multi-particle system, $N>1$, which is considerably more complicated [12]. There is now an effective interaction between the particles caused by the thermostat: if one particle increases (decreases) its speed due to the external field $\mathbf{E}$ the others have to decrease (increase) their speed to keep the total kinetic energy fixed. To study this system analytically we also make use of a stochastic version of the dynamics in which the collisions with the fixed obstacles are replaced by "virtual" collisions or scatterings [12]. These collisions, like the fixed scatterers, conserve energy and tend to make the angular distribution uniform.

We find analytically an autonomous equation for the N-particle speed distribution in the limit $\mathbf{E}\to 0$: exactly the same for the stochastic and deterministic models. This implies *ipso facto* that this distribution is the same for every chaotic (dispersing) billiard table and thus it is independent of the shape and position of the scatterers. It is a "universal" function whose exact shape in the NESS we determine explicitly. Using highly accurate numerical simulations the result seems to remain valid up to substantial values of $|\mathbf{E}|$. Just how large $\mathbf{E}$ can be depends on the shape of the table; see fig. 1. The new element in our analysis is the exploitation of a time scale separation which occurs for small $|\mathbf{E}|$.

We also find analytically the first order (in $\mathbf{E}$) correction to the invariant distribution which, unlike the speed distribution, depends on the shape of the table, but *only* through some properties which can be obtained from the $N=1$ solution of the problem. We have checked this expression numerically by computing, with high precision, the invariant measure for Table B in fig. 1. We also find analytically and verify numerically a simple explicit expression for the asymptotic $N\gg 1$ form of the speed distribution.

Analysis. – We shall consider first the stochastic model where the computations are simpler and essentially rigorous. The system now consists of N point particles in the unit 2D torus which move according to (1) between collisions (without the term $\mathcal{F}_i$). In addition each particle independently has a (virtual) collision with a Poisson rate equal to $\lambda(\mathbf{q}_i)|\mathbf{v}_i|$ for some position dependent rate $\lambda(\mathbf{q})>0$, *i.e.*, the weighted mean free path between collisions $\int_0^t\lambda(\mathbf{q}(t))|\dot{\mathbf{q}}(t)|\mathrm{d}t$ is an exponential random variable with mean one. The collision changes the angle which $\mathbf{v}$ makes with the x axis from θ' to θ according to some transition kernel $K(\theta,\theta')\,d\theta$. The exact form of K will turn out not to matter as long as $K(\theta,\theta')=K(\theta',\theta)$ and there is enough spreading to the direction of the velocity so that $\mathrm{d}\mathbf{q}\,\mathrm{d}\theta/2\pi$ is the unique invariant distribution for the system with one particle ($N=1$) and $E=0$. The scattering "closest" to that caused by collisions with fixed discs and the one we used in the simulations is the following: $\mathbf{v}'$ changes to $\mathbf{v}$ according to the rule

$$\mathbf{v}=\mathbf{v}'-2\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{v}'), \tag{3}$$

where $\hat{\mathbf{n}}$ is a unit vector in the direction of the momentum transfer from $\mathbf{v}'$ to $\mathbf{v}$. The direction of $\hat{\mathbf{n}}$ is chosen randomly with probability density $-(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}}')/2$, where $\hat{\mathbf{v}}'=\mathbf{v}'/|\mathbf{v}'|$, subject to the constraint $(\hat{\mathbf{n}}\cdot\mathbf{v}')<0$.

The "master" equation describing the time evolution of the N-particle velocity distribution function is, for the above rule, given by

see eq. (4) *above*

where $\mathbf{j}=\mathbf{J}/U$ as in (2), $\mathbf{Q}=(\mathbf{q}_1,\dots,\mathbf{q}_N)$, $\mathbf{V}=(\mathbf{v}_1,\dots,\mathbf{v}_i,\dots,\mathbf{v}_N)$, $\mathbf{V}'_i=(\mathbf{v}_1,\dots,\mathbf{v}'_i,\dots,\mathbf{v}_N)$, and $\mathbf{v}'_i$ is given in terms of $\mathbf{v}_i$ by (3). In the last term E is the magnitude of $\mathbf{E}$, *i.e.*, $\mathbf{E}=E\mathbf{e}$ for a unit vector $\mathbf{e}$.

Finally, $\mathcal{C}=\sum_{i=1}^{N}\mathcal{C}_i$ and $\mathcal{A}=\sum_{i=1}^{N}\mathcal{A}_i$ are the sums of collision and streaming terms which act independently and do not depend on E. We shall assume that $\lambda(\mathbf{q})$ is such that (4) has a unique steady-state solution for every $\mathbf{E}\neq 0$. A sufficient, but not necessary, condition is that $C^{-1}\leqslant\lambda(\mathbf{q})\leqslant C$ for some positive C.

Let us consider now what happens when E is small. We note first that when $E=0$ the speed of any particle does not change with time but the collisions (deterministic or stochastic) randomize the direction of the velocity of each particle. The distribution of speeds would then remain unchanged in time. When E is small the appropriate time scale for the change in the speed of the particle will be of order E^{-2}. Now on that time scale each particle will have undergone many collisions and so one may then assume that the direction of the velocity and the position of each particle will be uniformly distributed. We can thus expect to have an autonomous equation for the distribution of the speeds. Let us set $\mathbf{v}_i=r_i(\cos(\theta_i),\sin(\theta_i))$ where $r_i=|\mathbf{v}_i|$ and the angle θ_i is taken with respect to the field direction which we can assume is in the x-direction. Moreover, we set $\mathbf{R}=(r_1,\ldots,r_N)$ and $\mathbf{\Theta}=(\theta_1,\ldots,\theta_N)$. We then carry out a van Hove (weak-coupling) limit [15,16], rescaling the time by letting $t=\tau/E^2$ and set $\widetilde{W}(\mathbf{Q},\mathbf{V},\tau;E)=W(\mathbf{Q},\mathbf{V},tE^{-2};E)$. $\widetilde{W}$ satisfies the rescaled equation

$$\frac{\partial\widetilde{W}(\mathbf{Q},\mathbf{V},\tau;\mathbf{E})}{\partial\tau}= E^{-2}(\mathcal{A}+\mathcal{C})\widetilde{W}(\mathbf{Q},\mathbf{V},\tau;\mathbf{E})+E^{-1}\mathcal{B}\widetilde{W}(\mathbf{Q},\mathbf{V},\tau;\mathbf{E}) \tag{5}$$

We now assume that

$$\widetilde{W}(\mathbf{Q},\mathbf{V},\tau;E) = \widetilde{W}^{(0)}(\mathbf{Q},\mathbf{V},\tau)+E\widetilde{W}^{(1)}(\mathbf{Q},\mathbf{V},\tau) +E^2\widetilde{W}^{(2)}(\mathbf{Q},\mathbf{V},\tau)+o(E^2). \tag{6}$$

This is a very strong assumption, in fact stronger than necessary for the following conclusions. It will be better justified with a more detailed analysis [3].

Substituting (6) into (5) we get the following set of equations

$$0 = (\mathcal{A}+\mathcal{C})\widetilde{W}^{(0)}(\mathbf{Q},\mathbf{V},\tau) \tag{7}$$

$$0 = (\mathcal{A}+\mathcal{C})\widetilde{W}^{(1)}(\mathbf{Q},\mathbf{V},\tau) +\mathcal{B}\widetilde{W}^{(0)}(\mathbf{Q},\mathbf{V},\tau) \tag{8}$$

$$\frac{\partial\widetilde{W}^{(0)}(\mathbf{Q},\mathbf{V},\tau)}{\partial\tau} = (\mathcal{A}+\mathcal{C})\widetilde{W}^{(2)}(\mathbf{Q},\mathbf{V},\tau) +\mathcal{B}\widetilde{W}^{(1)}(\mathbf{Q},\mathbf{V},\tau). \tag{9}$$

Equation (7) implies that $\widetilde{W}^{(0)}(\mathbf{Q},\mathbf{V},\tau)$ depends only on $\mathbf{R}$, *i.e.*, $\widetilde{W}^{(0)}(\mathbf{Q},\mathbf{V},\tau)=\widetilde{W}^{(0)}(\mathbf{R},\tau)$. Since $\mathcal{B}F(\mathbf{V})$ is orthogonal to the functions that depend only on $\mathbf{R}$ if F depends only on $\mathbf{R}$, it follows that $\mathcal{E}_{\mathbf{Q},\mathbf{\Theta}}\mathcal{B}\widetilde{W}^{(0)}(\mathbf{R},\tau)=0$, where $\mathcal{E}_{\mathbf{Q},\mathbf{\Theta}}$ is the average on $\mathbf{Q}$ and $\mathbf{\Theta}$. Thanks to our hypotheses on λ and K we have that $(\mathcal{A}+\mathcal{C})F(\mathbf{Q},\mathbf{V})=0$ if and only if $F(\mathbf{Q},\mathbf{V})$ is a function of $\mathbf{R}$ alone so that $\widetilde{W}^{(1)}(\mathbf{Q},\mathbf{V},\tau)=-(\mathcal{A}+\mathcal{C})^{-1}\mathcal{B}\widetilde{W}^{(0)}(\mathbf{R},\tau)$ is well defined. We now insert this expression into (9) and average over $\mathbf{\Theta}$ and $\mathbf{Q}$. This does not effect the left-hand side since $\widetilde{W}^{(0)}$ depends only on $\mathbf{R}$ but it makes the first term on the right-hand side vanish leading to

$$\frac{\partial\widetilde{W}^{(0)}(\mathbf{R},\tau)}{\partial\tau}=-\mathcal{E}_{\mathbf{\Theta},\mathbf{Q}}\mathcal{B}(\mathcal{A}+\mathcal{C})^{-1}\mathcal{B}\widetilde{W}^{(0)}(\mathbf{R},\tau), \tag{10}$$

which is indeed an autonomous equation for $\widetilde{W}^{(0)}(\mathbf{R},\tau)$. It describes the effective dynamics on the scale τ in the limit $\mathbf{E}\to 0$, [15].

Equation (10) can be written out explicitly as

$$D^{-1}\frac{\partial\widetilde{W}^{(0)}(\mathbf{R},\tau)}{\partial\tau} = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial^2}{\partial r_i\partial r_j}\big[M_{ij}(\mathbf{R})\widetilde{W}^{(0)}(\mathbf{R},\tau)\big] +\sum_{i=1}^{N}\frac{\partial}{\partial r_i}\big[A_i(\mathbf{R})\widetilde{W}^{(0)}(\mathbf{R},\tau)\big], \tag{11}$$

where the components of the $N\times N$ matrix $\mathbf{M}$ are given by

$$M_{ij}(\mathbf{R})=\sum_{k=1}^{N}\frac{b_{ik}(\mathbf{R})b_{jk}(\mathbf{R})}{r_k}=\frac{1}{r_i}\delta_{ij}-\frac{r_i+r_j}{U}+\frac{r_ir_j}{U^2}\sum_{k=1}^{N}r_k \tag{12}$$

and

$$A_i(\mathbf{R})=-\frac{r_i}{U}\sum_{k=1}^{N}\frac{1}{r_k}+\frac{r_i}{U^2}\sum_{k=1}^{N}r_k,\qquad b_{ik}=\delta_{ik}-\frac{r_ir_k}{U}. \tag{13}$$

The diffusion constant D in (11) is just the integral of the velocity autocorrelation in the field direction $\mathbf{e}$ when the magnitude of the field $E=0$ and the speed is one, *i.e.*, $D=\mathbf{e}\cdot\mathbf{D}\mathbf{e}$, where

$$\mathbf{D}=\int_0^{\infty}\langle\mathbf{v}_1\otimes\mathbf{v}_1(t)\rangle\,\mathrm{d}t=\int_0^{\infty}\langle\mathbf{v}_1\otimes e^{(\mathcal{A}_1+\mathcal{C}_1)t}\mathbf{v}_1\rangle\,\mathrm{d}t, \tag{14}$$

and $\langle\cdot\rangle$ stands for averages with respect to the uniform measure $\mathrm{d}\mathbf{q}\,\mathrm{d}\theta/(2\pi)$ that is stationary for $E=0$.

D is in fact the only term in (11) which depends on the collision kernel $\mathcal{C}$ in (4). For a spatially uniform and isotropic scattering, *i.e.*, when $\lambda(\mathbf{q})$ is a constant and $K(\theta',\theta)=K(\theta'-\theta)$, we get $\mathbf{D}=D\mathbf{I}$ with

$$\begin{aligned} D &= \frac{1}{2\pi\lambda}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\infty}[\cos\theta\cos\theta(t)]\,\mathrm{d}t\\ &= \frac{1}{2\pi\lambda}\int_0^{\infty}\mathrm{d}t\int_0^{2\pi}\mathrm{d}\theta\cos(\theta)e^{t\mathcal{C}_1}\cos(\theta). \end{aligned} \tag{15}$$

For the specific model used in (4), $D=3/(4\lambda)$. In the case of the deterministic billiards D will depend on the shape of the table. Note, however, that the NESS corresponding to the stationary solution of (11) is independent of D which really just sets a time scale ($\tau \simeq t/(DE^2)$).

We note that

$$M = SS^* \qquad \text{with} \qquad S_{ij}(\mathbf{R}) = \frac{b_{ij}(\mathbf{R})}{\sqrt{r_j}} \tag{16}$$

which implies that (11) corresponds to a stochastic time evolution described by the Itô stochastic differential equation

$$\mathrm{d}r_i = -DA_i(\mathbf{R})\,\mathrm{d}t + \sum_{j=1}^{N} \sqrt{D}\sqrt{2}S_{ij}(\mathbf{R})\mathrm{d}B_i, \tag{17}$$

where B_i are N-independent Brownian motions. One can in fact first derive (17) and then obtain (11) [3].

Using some general theory [17], it follows from (11) that there is a unique solution of (11) which approaches, in the limit $\tau \to \infty$, a stationary state $\widetilde{W}^{(0)}(\mathbf{R})$. Let now $\widehat{W}(\mathbf{Q}, \mathbf{V}; \mathbf{E})$ be the stationary solution of (4) which also exists and is unique by the Döblin condition. Averaging $\widehat{W}(\mathbf{Q}, \mathbf{V}; \mathbf{E})$ on $\mathbf{Q}$ and $\mathbf{\Theta}$ to get $\widehat{W}(\mathbf{R}; \mathbf{E})$ and then taking the $\lim_{E\to 0} \widehat{W}(\mathbf{R}; \mathbf{E})$ we get the stationary solution of (11), $\widehat{W}^{(0)}(\mathbf{R})$ which coincides with $\widetilde{W}^{(0)}(\mathbf{R})$ and is independent of D. To compute it we observe that if $W(\mathbf{Q}, \mathbf{V}, t; \mathbf{E})$ solves (4) so does $W'(\mathbf{Q}, \mathbf{V}, t; \mathbf{E}) = h(U)W(\mathbf{Q}, \mathbf{V}, t; \mathbf{E})$ every positive function h. Moreover (4) is invariant under the rescaling

$$\mathbf{V} \to \rho\mathbf{V}, \qquad t \to \rho^{-1}t, \qquad \mathbf{E} \to \rho^2\mathbf{E}. \tag{18}$$

This suggests to look for $\widehat{W}_0$ of the form

$$\widehat{W}^{(0)}(\mathbf{R}) = h(U)F_0(\mathbf{R}), \tag{19}$$

where $F_0(\rho\mathbf{R}) = \rho^{2N-1}F_0(\mathbf{R})$ and $h(U)$ assures that $\widehat{W}_0$ has integral 1. With this assumption we get that F_0 satisfies the equation

$$\sum_{i=1}^{N} \left(\frac{1}{r_i}\frac{\partial^2 F_0}{\partial r_i^2} + \frac{2}{U}\frac{\partial F_0}{\partial r_i} \right) = 0. \tag{20}$$

This equation can be easily solved and we get, when the initial state is such that $U = N$,

$$\widehat{W}^{(0)}(\mathbf{R}) = \frac{1}{Z}\,\delta(U-N) \left[\sum_{i=1}^{N} r_i^3 \right]^{-\frac{2N-1}{3}} = \delta(U-N)\,F_0(\mathbf{R}), \tag{21}$$

where Z is just the normalization

$$Z = \int_{\sum r_i^2 = N} \left[\sum_{i=1}^{N} r_i^3 \right]^{-\frac{2N-1}{3}} \prod_{i=1}^{N} r_i\,\mathrm{d}r_i. \tag{22}$$

To get the one-particle marginal speed distribution $f_0(r; N)$ one has to integrate (21) over the variables $r_2, \ldots, r_N$. When $N \to \infty$ this yield the parameter-free universal distribution [12,18]

$$\lim_{N\to\infty} f_0(r; N) = C\exp(-cr^3), \tag{23}$$

where

$$C = \frac{3\Gamma\left(\frac{4}{3}\right)}{2\pi\Gamma\left(\frac{2}{3}\right)^2} \approx 0.2325, \qquad c = \left(\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \right)^{\frac{3}{2}} \approx 0.5355. \tag{24}$$

This of course is very far from any Maxwellian distribution.

Going beyond the limit $E \to 0$ we find the first-order correction (in E) to the stationary solution of (4):

$$\widehat{W}(\mathbf{R}, \mathbf{\Theta}; E) = \widehat{W}^{(0)}(\mathbf{R}) + E\widehat{W}^{(1)}(\mathbf{R}, \mathbf{\Theta}) + o(E), \tag{25}$$

where

$$\widehat{W}^{(1)}(\mathbf{R}, \mathbf{\Theta}) = (\mathcal{A} + \mathcal{C})^{-1}\mathcal{B}\widehat{W}^{(0)}(\mathbf{R}) = \delta(U-N)\,F_1(\mathbf{R}) \sum_{i=1}^{N} r_i c(\mathbf{q}_i, \theta_i) \tag{26}$$

with $F_1(\mathbf{R}) = (2N-1)\left[\sum_{i=1}^{N} r_i^3\right]^{-\frac{2N+2}{3}}$ and

$$c(\mathbf{q}_i, \theta_i) = \int_0^\infty e^{t(\mathcal{A}_i + \mathcal{C}_i)} \cos\theta_i \mathrm{d}t = -(\mathcal{A}_i + \mathcal{C}_i)^{-1} \cos\theta_i; \tag{27}$$

$\mathcal{C}_i$ and $\mathcal{A}_i$ are the collision and streaming operators defined on the right-hand side of (4). Note that, for $N = 1$, the invariant solution to (4) is simply

$$\widehat{W}(\mathbf{q}, \mathbf{v}; E) = \delta(|\mathbf{v}| - 1) \left(\frac{1}{2\pi r} + \frac{E}{r^3}c(\mathbf{q}, \theta) + o(E) \right) \tag{28}$$

so that $c(\mathbf{q}, \theta)$ is simply related to the $N = 1$ problem. The NESS current $\langle \mathbf{j} \rangle$ obtained from (11) is in agreement with that given by the Green-Kubo formula using $\widehat{W}^{(0)}$ as a reference measure.

Deterministic billiard. – The master (Liouville) equation for the deterministic model is given by

$$\frac{\partial W_d(\mathbf{Q}, \mathbf{V}, t)}{\partial t} = \mathcal{A}W_d + E\,\mathcal{B}W_d + \mathcal{C}_d W_d \tag{29}$$

with $\mathcal{C}_d W_d$ representing the collisions with the fixed convex obstacles. In this case, when $\mathbf{E} = 0$, rapid (exponential) approach to a stationary state which depends only on $\mathbf{R}$ (in the spatial domain $\mathcal{T}$ outside the obstacles) is due to the obstacle being convex [13]. In the presence of $\mathbf{E} \neq 0$, the stationary state is not absolutely continuous with respect to the Lebesgue measure, *i.e.*, it will not have a smooth density $\widehat{W}_d(\mathbf{Q}, \mathbf{V}; \mathbf{E})$, see [9,13]. On the other hand, starting with an initial smooth density $W_d(\mathbf{Q}, \mathbf{V}, 0; \mathbf{E})$, it will have a smooth density $W_d(\mathbf{Q}, \mathbf{V}, t; \mathbf{E})$ satisfying (29) for all finite time t. Let $\mu^t_{\mathbf{E}}(\mathrm{d}\mathbf{Q}, \mathrm{d}\mathbf{V}) = W_d(\mathbf{Q}, \mathbf{V}, t; \mathbf{E})\mathrm{d}\mathbf{Q}\,\mathrm{d}\mathbf{V}$ and set

$$\hat{\mu}_{\mathbf{E}}(\mathrm{d}\mathbf{Q}, \mathrm{d}\mathbf{V}) = \lim_{t\to\infty} \mu^t_{\mathbf{E}}(\mathrm{d}\mathbf{Q}, \mathrm{d}\mathbf{V}). \tag{30}$$

For $N = 1$ we know that $\hat{\mu}_{\mathbf{E}}(\mathrm{d}\mathbf{q}, \mathrm{d}\mathbf{v})$ exists and its projection on the energy surface $|\mathbf{v}| = 1$, can be written

$$\hat{\mu}_{\mathbf{E}}(\mathrm{d}\mathbf{Q},\mathrm{d}\mathbf{V}) = \delta(U-N)\left(F_0(\mathbf{R})\tilde{\mathrm{d}}\mathbf{Q}\,\mathrm{d}\mathbf{V} + EF_1(\mathbf{R})\sum_i r_i^2\delta\mu_1(\mathrm{d}\mathbf{q}_i,\mathrm{d}\theta_i)\tilde{\mathrm{d}}\mathbf{Q}^i\,\mathrm{d}\mathbf{V}^i + o(E)\right), \tag{31}$$

as $\hat{\mu}_{\mathbf{E}}(\mathrm{d}\mathbf{q},\mathrm{d}\theta) = \mathrm{d}\mathbf{q}\,\mathrm{d}\theta/(2\pi) + E\delta\mu_1(\mathrm{d}\mathbf{q},\mathrm{d}\theta) + o(E)$, with $\delta\mu_1$ singular with respect to $\mathrm{d}\mathbf{q}\mathrm{d}\theta$. We note that the property (18) remains true for any solution $W_d(\mathbf{Q},\mathbf{V},t;\mathbf{E})$ of (29). The expansion (25) can be generalized to the deterministic billiard by replacing $c(\mathbf{q},\theta)$ with $\delta\mu_1(\mathrm{d}\mathbf{q},\mathrm{d}\theta)$. More precisely we have

see eq. (31) *above*

where $\tilde{\mathrm{d}}\mathbf{Q} = \prod_i \tilde{\mathrm{d}}\mathbf{q}_i$, with $\tilde{\mathrm{d}}\mathbf{q}$ the normalized restriction of the Lebesgue measure to $\mathbb{T}^2\backslash$obstacles, $\mathrm{d}\mathbf{Q}^i = \prod_{j\neq i}\tilde{\mathrm{d}}\mathbf{q}_j$, $\mathrm{d}\mathbf{V}^i = \prod_{j\neq i}\mathrm{d}\mathbf{v}_j$ and $F_1(\mathbf{R})$ is defined after (26). The above expression implies that $\hat{\mu}_N(\mathrm{d}\mathbf{Q},\mathrm{d}\mathbf{V}) = \lim_{\mathbf{E}\to 0}\hat{\mu}_{\mathbf{E},N}(\mathrm{d}\mathbf{Q},\mathrm{d}\mathbf{V})$ is absolutely continuous with respect to the Lebesgue measure on the energy sphere and depends only on the speeds. We cannot prove this statement rigorously but it is well verified by our numerical simulations involving the full billiard table or just a portion of it.

Numerical results. – We have concentrated our numerical simulation on the deterministic billiard system using the billiard tables depicted in fig. 1. Table A is the same used in [12]. Table B has the central obstacle moved down to break the symmetry but remaining rather close to Table A. Table C instead was chosen to be as asymmetric and as far from Table A as possible.

We computed the one-particle marginal of the speed distribution for all 3 tables. To obtain an accurate and reliable result we ran a very long trajectory recording the speed of particle 1 every time of the order of E^{-2}. In this way we can assume that the data we collected form a random sample from the distribution $f_{0,d}(r,N)$. This allows us to use the Kolmogorov-Smirnov test to check whether $f_{0,d}(r,N) = f_0(r,N)$ described after (21), see [19,20].

In fig. 2 we plot the marginal distribution for all three tables when $\mathbf{E} = 0.015625(\cos(\phi),\sin(\phi))$ with $\phi = \pi/2$ and $N = 512$ together with the theoretical prediction coming from (23). The P-value of the KS test for the cases shown in these figures was greater than 23% giving a strong evidence that our hypothesis on the distribution of the observed data is indeed correct. A more extensive report on these simulations can be found at `http://www.math.uab.edu/~khu/g/gt/speed.html`.

To check whether the full distribution in (31) is valid for the deterministic billiard we chose $N = 3$ and fixed $U = 3$ so that we can take $r_3^2 = 3 - r_1^2 - r_2^2$. Fixing $\delta_r = \sqrt{3}/30$, let $\Delta_{i,j}(r_1,r_2)$ be the characteristic function of the square of side δ_r centered at $((i+0.5)\delta_r,(j+0.5)\delta_r)$. We ran 1000 trajectories of average length $2\cdot 10^8$ system time units for Table B with $\mathbf{E} = 0.04(\cos(\phi),\sin(\phi))$ with $\phi = \pi/3$. The results are in good agreement

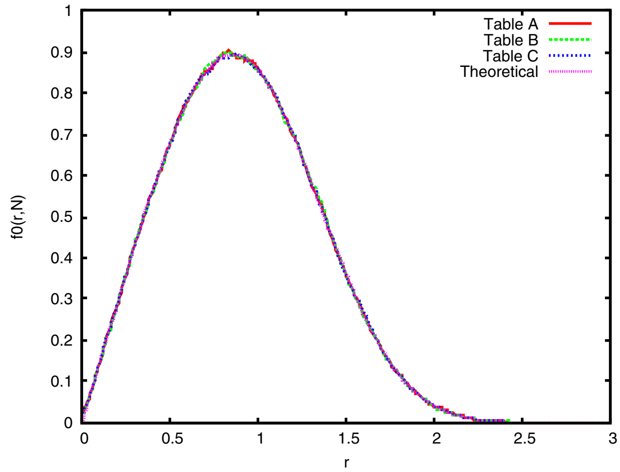

Fig. 2: (Colour on-line) Comparison between the one-particle speed marginals of the three different tables and also (23) with 512 particles.

with the prediction of (11). See `http://www.math.uab.edu/~khu/g/gt/speed.html`, where plots of the results of these simulations can be found.

Concluding remarks. – Our main results are the derivation of the universal equations (11), (21) and (26) and the verification of the latter by very extensive controlled numerical simulations. We believe that the multi-scale analysis of this model system will find many applications in the study of nonequilibrium systems. In particular, the equivalence of the dynamics to that of a stochastic differential equation is similar to that obtained in [21].

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