Commun. Math. Phys. 189, 263 – 275 (1997)

Communications in
Mathematical Physics

Reversibility, Coarse Graining and the Chaoticity Principle

F. Bonetto[1], **G. Gallavotti**[2]

[1] Matematica, Università di Roma “La Sapienza”, P.le Moro 2, 00185 Roma, Italia.
E-mail: giovanni@ipparco.roma1.infn.it
[2] Fisica, Università di Roma “La Sapienza”, P.le Moro 2, 00185, Roma, Italia.
E-mail: bonetto@onsager.rutgers.edu

Received: 28 February 1996 / Accepted: 12 February 1997

Dedicated to the memory of Roland Dobrushin

Abstract: We describe a way of interpreting the chaotic principle of [GC1] more extensively than it was meant in the original works. Mathematically the analysis is based on the dynamical notions of Axiom A and Axiom B and on the notion of Axiom C, that we introduce arguing that it is suggested by the results of an experiment ([BGG]) on chaotic motions. Physically we interpret a breakdown of the Anosov property of a time reversible attractor (replaced, as a control parameter changes, by an Axiom A property) as a spontaneous breakdown of the time reversal symmetry: the relation between time reversal and the symmetry that remains after the breakdown is analogous to the breakdown of T-invariance while TCP still holds.

1. Introduction

In reference [GC2] a general mechanical system in a non equilibrium situation was considered. Calling $\mathcal{C}$ the (compact or “finite”) phase space of the “observed events”, μ_0 the volume measure on it and S the map describing the time evolution (regarded as a discrete invertible mapping of “observed events” into the “next ones”, *i.e.* as a Poincaré map on some surface in the full phase space) a principle holding when motions have an empirically chaotic nature was introduced:

1. Chaotic hypothesis. A chaotic many particle system in a stationary state can be regarded, for the purpose of computing macroscopic properties, as a smooth dynamical system with a transitive Axiom A globally attracting set. In reversible systems it can be regarded, for the same purposes, as a smooth transitive Anosov system.

• *Chaotic* is an empirical qualitative notion that means that most points of the attracting set have a stable and an unstable manifold with positive dimension. In the applications in [GC1, GC2, G1] the use of the hypothesis, which is a natural extension of a principle proposed by Ruelle, was based on reversibility and on transitivity.

• An *attracting set* is a closed invariant set such that all points in its vicinity evolve (in the future) tending to it and such that no subset has the same property (*i.e.* it is "minimal"). A set is *globally attracting* if it is attracting and *all* points of an open dense set evolve tending to it.
• An *Axiom A* attracting set is a hyperbolic attracting set: *i.e.* an attracting set with each of its points possessing stable and unstable manifolds depending continuously upon the points and with contraction and expansion rates bounded uniformly away from 0. Furthermore the periodic points are dense.
• *Reversibility* means that there is a smooth map i of $\mathcal{C}$ onto itself that changes the sign to time in the sense that $iS = S^{-1}i$ and $i^2 = 1$. As is well known reversibility should not be confused with the invertibility of the map S (always assumed below).
• *Transitivity* is intended to mean that the stable and unstable manifolds of the attracting set points are dense on it (this is not a very strong requirement in view of (7.6) in [Sm], p. 783).
• *Anosov system* is a dynamical system in which the whole phase space is hyperbolic.

Note that if the Axiom A attracting set is supposed to be also a smooth manifold then the restriction of the dynamics to it is an Anosov system. This is the meaning that we give, in this paper, to the second assumption in the above hypothesis (see Sect. 6 of [BGG]).

However in the previous paper [G4] transitivity was instead intended to mean, at least in the reversible cases, density of the stable and unstable manifolds of the attracting set points on the entire phase space (so that the system was in fact a transitive Anosov system). This is not always a property that one may be willing to consider as reasonable.

It is reasonable for systems that are very close to conservative ones (as in [G4]). But it is very likely (see [BGG], Sect. 6, Fig. 14) to be incorrect in systems that are under strong non conservative forces, even if still evolving with a reversible dynamics. In fact the attracting sets of such systems often evolve, as the strength of the forces increases, from a very chaotic initial attracting set to a more ordered situation characterized by a periodic orbit or by a very small attracting "tube" almost identical to a periodic orbit; the evolution shows a gradual decrease of the dimension and of the phase space region occupied by the attracting set, which therefore quite soon may become contained in a proper closed subset of phase space (so that the system cannot have stable and unstable manifolds with dimension half that of phase space: a necessary consequence of time reversibility in transitive Anosov systems).

In such cases it is still reasonable, see [R1, ER], to think that the attracting set, if chaotic, can be regarded "just" as an *Axiom A attracting set*, an assumption weaker than assuming that the system is an Anosov system.

What can be said for such systems? Is there a suitable reformulation of the chaoticity principle that could make it applicable even in very strongly forced (but still "chaotic") systems making possible an analysis similar to that leading to the fluctuation theorem of [GC1, GC2] or to the Onsager relations of [G4]? These are the questions we address here.

2. A Distinction Between Anosov and Axiom A Properties. Statistics. Axioms B and C

We shall try to adopt the notations used in the well known paper by Smale, [Sm]. Furthermore in this paper, as in [GC1, GC2], a distinction will be made between an *attractor* and its closure that we call more properly an *attracting set*: this is often a rather confusing point because some authors identify an attractor with its closure. Here we shall not. Thus, recalling that μ_0 denotes the volume measure on $\mathcal{C}$, we shall formally say that:

Definition 1. *A point $x \in \mathcal{C}$ admits a statistics if there is a probability distribution in phase space such that:*

$$\lim_{M\to\infty} \frac{1}{M} \sum_{j=0}^{M-1} F(S^j x) = \int_{\mathcal{C}} F(y)\mu(dy) \tag{2.1}$$

for all continuous functions F on phase space $\mathcal{C}$.,

and:

Definition 2. *An attractor is a set C enjoying the properties: (a) it is invariant and dense in an attracting set, (b) μ_0–almost all points in its vicinity admit the same statistics μ and $\mu(C) = 1$, (c) it has minimal Hausdorff dimension and (d) it is minimal.*[1] *An attracting set verifies Axiom A if it has the properties: (i) each of its points is hyperbolic, (ii) the periodic points are dense on it, (iii) it contains a dense orbit.*

With this definition one has to live with the fact that there will be several essentially identical attractors: if the distribution μ gives, for instance, 0 probability to individual points on the attractor C then any subset of C obtained by removing the orbis of a countable number of points will still be an attractor (of course with the same closure as C itself). This is the main reason why it is wise to distinguish the notions of attractor and of attracting set; in the cases met in this paper it will be appropriate to call the latter an *attracting basic set* (see below).

It is useful to recall some more general definitions and properties:

- a dynamical system *verifies Axiom A* if each point in the set Ω of "nonwandering points" (*i.e.* in the set of all "recurrent points") is "hyperbolic", *i.e.* each nonwandering point admits stable and unstable manifolds, continuously dependent on the point and with expansion and contraction rates uniformly bounded away from 0; furthermore the periodic points are dense on Ω, [Sm], p.777. The closure of an attractor for an Axiom A system is an Axiom A attracting set in the above sense.
- Axiom A systems have a rather simple structure as the sets of their nonwandering points consist of a finite number of closed invariant indecomposable topologically transitive sets, called *basic sets*, see [Sm], p.777.[2]
- A basic set that is the closure of an attractor is called an attracting basic set, likewise one defines a repelling basic set.[3]
- Another interesting class of dynamical systems is the class of *Axiom B systems* (see [Sm.] p 778): they are the dynamical systems verifying Axiom A and the further mild transversality property[4] that if Ω_i, Ω_j are a pair of basic sets such that the stable set

[1] This means that any other set with the same properties has a closure that contains the closure clos(C).

[2] Topologically transitive, [Sm] p.776, means that they contain a point with a dense orbit. Note that this is weaker than transitive in the sense of Sect. 1. Indecomposable means that they contain no subset with the same properties. The basic sets are the *building* pieces (or the "bases") of the part of phase space where the dynamics is non trivial. Transitive Anosov systems are simply Axiom A systems with a (unique) basic set coinciding with the whole phase space.

[3] Thus a basic set for an Axiom A system can be regarded as a dynamical system in itself: in this case it may fail to be Anosov only because in general it is not a smooth manifold but just a closed set.

[4] Mild because of Theorem (6.7), p. 779, in [Sm].

$W^s(\Omega_i)$ and the unstable set $W^u(\Omega_j)$ intersect *then* they intersect transversally (see below).

• The stable (resp. unstable) set of Ω_i (resp. Ω_j) is the union of the stable (resp. unstable) manifolds of all its points (see [Sm], p. 777). The intersection between $W^s(\Omega_i)$ and $W^u(\Omega_j)$ is *transversal* if the stable manifold $W^s(p)$ and the unstable manifold $W^u(q)$ of any two *periodic points* $p \in \Omega_i$ and $q \in \Omega_j$ have a point of transversal intersection (see [Sm], p. 783). Finally two manifolds intersect transversally at a point if their tangent planes span the full tangent plane (see [Sm], p. 752).

We also need to recall that, particularly in the numerical experiments, for dynamical systems with "chaotic behavior" it is usually also assumed that, after the "obvious" conservation laws and symmetries are taken into account, the *whole phase space* admits a statistics, in the sense that almost all points (with respect to the volume measure) admit a statistics, the same for all of them. This is usually called the *zero*$^{\text{th}}$ *law*, [UF]:

Extended zero$^{\text{th}}$ law: *A dynamical system $(\mathcal{C}, S)$ describing a many particle system (or a continuum such as a fluid) generates motions that admit a statistics μ in the sense that, given any (smooth) macroscopic observable F defined on the points x of the phase space $\mathcal{C}$, the time average of F exists for all μ_0–randomly–chosen initial data x and is given by:*

$$\lim_{M\to\infty} \frac{1}{M} \sum_{k=0}^{M-1} F(S^j x) = \int_{\mathcal{C}} \mu(dx')F(x') \tag{2.2}$$

with μ being a S–invariant probability distribution on $\mathcal{C}$.

It is important to note the physical meaning of the above law: in fact, among other things, it implies that the dynamical system $(\mathcal{C}, S)$ *cannot have more than one attracting basic set*. We shall say that a system for which the property described by the above "law" holds *verifies the zero*$^{\text{th}}$ *law*.

For our purposes only systems verifying at least Axiom B will be relevant. For such systems the notion of *diagram* of a dynamical system, see [Sm] p. 754, allows us to interpret the zero$^{\text{th}}$ law as saying that the diagram of the system is a partially ordered set with unique top and bottom points.

In this paper we shall deal with reversible dynamical systems $(\mathcal{C}, S)$ verifying Axiom B and the zero$^{\text{th}}$: hence with a unique attracting basic set Ω_+ and a unique repelling basic set Ω_-. Furthermore the attracting set will be assumed transitive in the sense that the stable and unstable manifolds of each of its points are dense on it.

The Axiom B and the zero$^{\text{th}}$ law could be reasonably taken as definitions of models for globally "chaotic" or "globally hyperbolic systems".

However the problem that we pose in the next section suggests that the appropriate notion for "globally hyperbolic" or "globally chaotic" dynamical systems is somewhat stronger. Its definition has been suggested by our effort to interpret the results of the experiment [BGG] and we allowed ourselves to give the name of *Axiom C systems* to systems verifying a stronger property; to describe it we introduce the notion of distance of a point x to the basic sets $\{\Omega_i\}$ of an Axiom A system as:

$$\delta(x) = \min\left\{\min_i \frac{d_{\Omega_i}(x)}{d_0}, \min_{j,\, -\infty<n<+\infty} \frac{d_{\Omega_j}(S^n x)}{d_0}\right\}, \tag{2.3}$$

where d_0 is the diameter of the phase space $\mathcal{C}$, $d_{\Omega_i}(x)$ is the distance of the point x from the basic set Ω_i, the minimum over i runs over the attracting or repelling basic sets and the minimum over j runs over the other $k \ge 0$ basic sets.

We can then define the *Axiom C* systems as:

Definition 3. *A smooth dynamical system $(\mathcal{C}, S)$ verifies Axiom C if it is Anosov or if it verifies Axiom A and:*
(1) among the basic sets there are a unique attracting and a unique repelling basic sets, denoted Ω_+, Ω_- respectively, with (open) full volume dense basins that we call the poles of the system (future or attracting and past or repelling poles, respectively).
(2) for every $x \in \mathcal{C}$ the tangent space T_x admits a Hölder–continuous[5] decomposition as a direct sum of three subspaces T_x^u, T_x^s, T_x^m such that:

$$
\begin{array}{llll}
a) & dS\,T_x^\alpha & = T_{Sx}^\alpha\,, & \alpha = u, s, m\,,\\
b) & |dS^n w| & \le Ce^{-\lambda n}|w|\,, & w \in T_x^s,\ n \ge 0,\\
c) & |dS^{-n} w| & \le Ce^{-\lambda n}|w|\,, & w \in T_x^u,\ n \ge 0,\\
d) & |dS^n w| & \le C\delta(x)^{-1}e^{-\lambda |n|}|w|, & w \in T_x^m\ \forall n,
\end{array}
$$

where the dimensions of T_x^u, T_x^s, T_x^m are > 0 and $\delta(x)$ is defined in (2.3).
(3) if x is on the attracting basic set Ω_+ then $T_x^s \oplus T_x^m$ is tangent to the stable manifold in x; viceversa if x is on the repelling basic set Ω_- then $T_x^u \oplus T_x^m$ is tangent to the unstable manifold in x.

Although T_x^u and T_x^s are not uniquely determined the planes $T_x^s \oplus T_x^m$ and $T_x^u \oplus T_x^m$ are uniquely determined for $x \in \Omega_+$ and, respectively, $x \in \Omega_-$.

It is clear that an Axiom C system is necessarily also an Axiom B system verifying the zero$^{\text{th}}$ law (as it follows from [R2]). We do not know an example of an Axiom B system with a unique attracting and a unique repelling basic set which is not at the same time an Axiom C system.

Apart from property (1) that is meant to imply the validity of the zero$^{\text{th}}$ law, one can also say that ("at most") the real difference between an Axiom B and an Axiom C system is that the latter has a stronger, and more global, hyperbolicity property.

Namely, if Ω_+ and Ω_- are the two poles of the system the stable manifold of a periodic point $p \in \Omega_+$ and the unstable manifold of a periodic point $q \in \Omega_-$ not only have a point of transversal intersection, but they intersect transversally *all the way* on a manifold connecting Ω_+ to Ω_-; the unstable manifold of a point in Ω_- will accumulate on Ω_+ *without winding around it*.

In fact one can "attach" to $W^s(p)$, $p \in \Omega_+$, points on Ω_- as follows: we say that a point $z \in \Omega_-$ is *attached* to $W^s(p)$ if it is an accumulation point for $W^s(p)$ and there is a curve *with finite length* linking a point $z_0 \in W^s(p)$ to z and *entirely lying on* $W^s(p)$, with the exeption of the endpoint z. A drawing helps understanding this simple geometrical construction, slightly unusual because of the density of $W^s(p)$ on Ω_-,.

We call $\overline{W}^s(p)$ the set of the points *either on* $W^s(p)$ *or just attached to* $W^s(p)$ on the system basic sets (the set $\overline{W}^s(p)$ should not be confused with the closure clos($W^s(p)$), which is the whole space, see [Sm], p. 783). If a system verifies Axiom C the set $\overline{W}^s(p)$ intersects Ω_- on a stable manifold, by 2) in the above definition.

The definition of $\overline{W}^u(q)$, $q \in \Omega_-$, is defined symmetrically by exchanging Ω_+ with Ω_-. Furthermore if a system verifies Axiom C and $p \in \Omega_+$, $q \in \Omega_-$ are two periodic points, on the attracting basic set and on the repelling basic set of the system respectively,

[5] One might prefer to require real smoothness, *e.g.* C^p with $1 \le p \le \infty$: but this would be too much for rather trivial reasons. On the other hand Hölder continuity might be equivalent to simple C^0–continuity as in the case of Anosov systems, see [AA, Sm].

then $\overline{W}^s(p)$ and $\overline{W}^u(q)$ have a dense set of points in Ω_+ and Ω_-, respectively. Note that $\overline{W}^s(p) \cap \overline{W}^u(q)$ is dense in $\mathcal{C}$ as well as in Ω_+ and Ω_-. This follows from the density of $W^s(p)$ and $W^u(q)$ on Ω_+ and Ω_- respectively and from the continuity of T_x^m. Furthermore if $z \in \Omega_+$ is such that $z \in \overline{W}^s(p) \cap \overline{W}^u(q)$ then the surface $\overline{W}^s(p) \cap \overline{W}^u(q)$ intersects Ω_- in a unique point $\tilde{z} \equiv \tilde{\imath}z$ which can be reached by the shortest smooth path on $\overline{W}^s(p) \cap \overline{W}^u(q)$ linking z to Ω_- (the path is on the surface obtained as the envelope of the tangent planes T^m, but it is in general not unique even if T_m has dimension 1, see the example in Sect. 4 below).

The map $\tilde{\imath}$, as a map of $\Omega_+ \cup \Omega_-$ into itself, commutes with both S and i, squares to the identity and will play a key role in the following analysis.

We conjecture that the Axiom C systems are Ω–stable in the sense of Smale, [Sm] p. 749 (*added on revision: this follows from Robbin's theorem*, see [R3], p. 170).

3. Axiom A, B, C and Time Reversibility: The Problem

If one considers the closure $\Omega_+ = \text{clos}(C)$ of an attractor C verifying Axiom A then the action of the dynamics S on it fails to be an Anosov system only because clos(C) *might* be a fractal set rather than a smooth surface.

In nonequilibrium statistical mechanics the dimensionality of the attractors is usually very large so that their fractality is likely to be irrelevant. This is part of the hypothesis that the system can be regarded as an Anosov system *for the purpose of studying averages of relevant quantities*. And in fact the Anosov property is used in the above references only to obtain a representation of the SRB distribution, *i.e.* of the distribution describing the averages of observables.

The same representation holds for the SRB distribution on an Axiom A attracting set. For this reason the fractality of an attractor was regarded in [GC2] as "an unfortunate accident".

Therefore the *really non trivial hypothesis* in the mentioned applications is the *reversibility of the motion on the attracting set*. Such reversibility is of course implied by the reversibility of the motion on the whole phase space if the attracting set and the whole phase space coincide: in the above references this was taken as a consequence of the chaoticity hypothesis.

However one may wish to see how far this is justified in the cases in which the attractor C is really smaller than the whole phase space. We shall refer to such cases as the cases in which the attracting set verifies Axiom A: we therefore include under the latter denomination also the case in which the attracting set is a smooth surface (and could therefore be said to be an Anosov system). The possible fractality of the closure of the attractor or its smoothness play no role in the following.

Suppose that the *reversible* mechanical system under consideration verifies Axiom C (a stronger notion than Axiom B, and a kind of "global hyperbolicity" condition as discussed in Sect. 2). Suppose that the attracting pole $\Omega_+ = \text{clos}(C)$ *is not* the whole phase space $\mathcal{C}$. Can one then conclude that the fluctuation theorem of [GC1] holds? or at least some modification of it?

We "answer" this in the affirmative by noting that the global time reversal map i, *a priori* assumed to exist, induces on the pole $\Omega_+ = \text{clos}(C)$ of the system a natural "smooth" (*i.e.* Hölder continuous) map i^* which verifies:

$$i^* S = S^{-1} i^*, \qquad (i^*)^2 = 1. \tag{3.1}$$

In fact since the map $\tilde{\imath}$ commutes with S and maps the attracting pole Ω_+ onto the repelling pole Ω_- we can set $i^* = \tilde{\imath}\, i$ and define a map of $\Omega_\pm$ into themselves verifying:

$$i^* S = \tilde{\imath}\, i\, S = \tilde{\imath}\, S^{-1}\, i = S^{-1}\, i^*. \tag{3.2}$$

Hence i^* is a time reversal on the future attracting set. *Note that i^* is not the restriction of i to the future attracting set*. This map will be the *local time reversal*. One should stress that since $\tilde{\imath}$ is defined only on $\Omega_\pm$ also i^* has *only* a meaning *as a map of such sets onto themselves*.

Its existence immediately implies the validity of a fluctuation theorem ([GC1, GC2, G3]) for systems that are globally reversible and chaotic in the sense that they verify Axiom C (and hence verify the zero$^{\text{th}}$ law and have an Axiom A attracting set), with the only difference that the theorem applies to the phase space contraction that occur on Ω_+ rather than to the phase space contraction occurring in the whole phase space $\mathcal{C}$ (see Sect. 6, (b), for a discussion). Moreover it implies also that the chaotic hypothesis can be conceptually simplified. Curiously enough this does not even require a modification of its formulation, but it allows for a broader intrepretation of it as the word "Anosov" can be essentially replaced by "reversible Axiom A attracting set" and this covers explicitly the cases in which the attracting set is not dense on phase space.

It is important to note here that the existence of i^* is not trivial: because i^* *cannot be* (unless $\mathcal{C} = \Omega_+$, *i.e.* unless the future pole of the system is the whole phase space so that the system is actually an Anosov system) the restriction to Ω_+ of the time reversal map i, as one would naively surmise.

In fact reversible systems with Axiom A attractors usually have attractors C_+ and C_- for the forward motion and for the backward motions which are *distinct*, [S1], in the sense that $\Omega_+ \cap \Omega_- \equiv \text{clos}(C_+) \cap \text{clos}(C_-) = \emptyset$. In such cases the time reversal i maps Ω_+ into Ω_- and viceversa. Therefore although the global time reversal i has the property $Si = iS^{-1}$ it does not leave invariant the attracting basic set $\Omega_+ = \text{clos}(C_+)$.

The map i^* is an *effective time reversal* acting on the closure of the attractor $\Omega_+ = \text{clos}(C_+)$ for the future motion. Its existence could be expected on philosophical grounds: if a system is reversible there should be no way of knowing whether one is moving on the *future pole* Ω_+ or on the *past* pole Ω_-. In particular we should be able to see that the motion is reversible without ever even knowing about the existence of the past attractor C_-. Hence we expect that there is a "*local time reversal*" i^* on both the future and the past attractors: and the problem is to find a way to construct, at least in principle, i^*.

The reader will notice the analogy between the above picture and the spontaneous symmetry breaking: when the attractor dimension decreases (because some Lyapunov exponent changes sign as a parameter changes) the time reversal symmetry is no longer valid, but some other symmetry survives which still has the effect of changing the sign to time: a well known example is the breaking of T-symmetry in relativistic quantum mechanics, with the TCP-symmetry remaining valid.

In Sect. 4 we present a model in which i^* can be constructed and provides the paradigm of a reversible Axiom C system. In Sect. 5 we discuss the meaning in symbolic dynamics of the map i^*. On the mathematical side there are various points that would require closer investigation. But the discussion seems to indicate that the scenario for the construction of i^* should work rather generally, as we think it is quite naturally suggested by the results of the experiment in [BGG].

4. An Example

We give here an example in which i^*, the local time reversal, arising in the applications can be easily constructed. The example illustrates what we think is a typical situation. The poles $\Omega_\pm$, closures of an attractor C_+ and respectively of a repeller C_-, will be two compact regular surfaces, identical in the sense that they will be mapped into each other by the time reversal i defined below.

If x is a point in $M_* = \Omega_+ = \text{clos}(C_+)$ the generic point of the phase space will be determined by a pair (x, z), where $x \in M_*$ and z is a set of transversal coordinates that tell us how far we are from the attractor. The coordinate z takes two well defined values on Ω_+ and Ω_- that we can denote z_+ and z_- respectively.

The coordinate x identifies a point on the compact manifold M_* on which a reversible transitive Anosov map S_* acts (see [G3]). And the map S on phase space is defined by:

$$S(x, z) = (S_* x, \tilde{S} z), \tag{4.1}$$

where $\tilde{S}$ is a map acting on the z coordinate (marking a point on a compact manifold) which is an evolution leading from an unstable fixed point z_- to a stable fixed point z_+. For instance z could consist of a pair of coordinates v, w with $v^2 + w^2 = 1$ (*i.e.* z is a point on a circle) and an evolution of v, w could be governed by the equation $\dot{v} = -\alpha v,\ \dot{w} = E - \alpha w$ with $\alpha = Ew$. If we set $\tilde{S}z$ to be the time 1 evolution (under the latter differential equations) of $z = (v, w)$ we see that such evolution sends $v \to 0$ and $w \to \pm 1$ as $t \to \pm\infty$ and the latter are non marginal fixed points for $\tilde{S}$.

Thus if we set $S(x, z) = (S_* x, \tilde{S} z)$ we see that our system is hyperbolic on the basic sets $\Omega_\pm = M_* \times \{z_\pm\}$ and the future pole $\Omega_+ = \text{clos}(C_+)$ is the set of points (x, z_+) with $x \in M_*$; while the past pole $\Omega_- = \text{clos}(C_-)$ is the set of points (x, z_-) with $x \in M_*$.

Clearly the two poles are mapped into each other by the map $i(x, z_\pm) = (i^* x, z_\mp)$. But on each attractor a "local time reversal" acts: namely the map $i^*(x, z_\pm) = (i^* x, z_\pm)$.

The system is "chaotic" as it has an Axiom A attracting set with closure consisting of the points having the form (x, z_+) for the motion towards the future and a different Axiom A attracting set with closure consisting of the points having the form (x, z_-) for the motion towards the past. In fact the dynamical systems (Ω_+, S) and (Ω_-, S) obtained by restricting S to the future or past attracting sets are Anosov systems because $\Omega_\pm$ are regular manifolds.

We may think that in the reversible cases the situation is always the above: namely there is an "irrelevant" set of coordinates z that describes the departure from the future and past attractors. The future and past attractors are copies (via the global time reversal i) of each other and on each of them is defined a map i^* which inverts the time arrow, *leaving the attractor invariant*: such map will be naturally called the *local time reversal*.

In the above case the map i^* and the coordinates (x, z) are "obvious". The problem is to see that they are defined quite generally under the only assumption that the system is reversible and has unique future and unique past attractors that verify the Axiom A. This is a problem that is naturally solved in general when the system verifies the Axiom C of Sect. 2 (see Sect. 3, (3.1) above).

In the following section we shall describe the interpretation of i^* in terms of symbolic dynamics when the system verifies Axiom C: as one may expect the construction is simple but it is deeply related to the properties of hyperbolic systems such as their Markov partitions.

5. Local Time Reversal and Markov Partitions

In this section we discuss the properties of the map i^* and its relation with the Markov partitions and the symbolic dynamics.

We assume the reader is familiar with the notion of Markov partition: in any event the results of this paper logically follow those of [GC1, GC2 and G2, G3] and we can expect that only readers familiar with those papers can have any interest in the present one.

In [GC1] we mention that a transitive Anosov reversible system admits a Markov partition $\mathcal{P} = \{Q_\sigma\}$, which is invariant under time reversal: $i\mathcal{P} = \mathcal{P}$. This means that for every element $Q \in \mathcal{P}$ one can find an element $Q' \in \mathcal{P}$ such that $iQ = Q'$.

If the dynamical system only has a transitive Axiom A attracting set we can still construct a Markov partition $\mathcal{P} = i\mathcal{P}$ but it will not have a transitive transition matrix. The transition matrix is in fact defined by setting $T_{\sigma,\sigma'} = 1$ if $SQ_\sigma \cap \text{int}(Q_{\sigma'}) \neq \emptyset$ (here $\text{int}(Q)$ are the interior points of Q *relatively* to the closure of the attractor) and $T_{\sigma,\sigma'} = 0$ otherwise. And transitivity means that there is a power k of T such that $T^k_{\sigma,\sigma'} > 0$ for all σ, σ' (see comments after the *chaotic hypothesis* Sect. 1 and footnote 2).

The lack of transitivity in the above sense is simply due to the fact that the Markov partition $\mathcal{P}$ really splits into two transitive Markov partitions, one, denoted $\mathcal{P}_+$, paving the closure $\Omega_+ = \text{clos}(C_+)$ of the future attractor and one, denoted $\mathcal{P}_-$, paving the closure $\Omega_- = \text{clos}(C_-)$ of the past attractor C_-. And of course there is no possibility of a transition from one to the other under the action of S as the two are S-invariant sets.

But for Axiom C systems the Markov partition can be built in a special way that takes into account more deeply the global time reversal symmetry of the system. Let in fact O be a fixed point of S on Ω_+ (if no fixed point exists O can, for the purposes of the following discussion, be replaced by a point on one of the periodic orbits on Ω_+; recall that by the Axiom A property the periodic orbits are dense on $\Omega_\pm$).

We shall assume, for simplicity, that Ω_+ (hence Ω_-) are smooth surfaces: then we consider the stable manifold of O. The latter is dense on Ω_+ because of the assumed transitivity of the attractor and it has a part that is not contained on the attracting set (because we are supposing that the attractor is *not* dense in phase space).

If n_0 is the dimension of the pole Ω_+ and n_s is the dimension of the part of the stable manifold W^s_O lying on Ω_+ then the dimension of the stable manifold will be $n_s + n$ for some $n \geq 1$. The manifold W^s_O will intersect the pole Ω_-: otherwise it would lead to another repelling basic set, violating the assumption that there are only two poles (*i.e.* only one attractor for the future motion and one for the backward motion as expressed by the zero$^{\text{th}}$ law above, see 3) in the definition of Axiom C).

The pole Ω_- has (by the time reversal symmetry) the same dimension n_0 of the pole Ω_+ and its intersection with $\overline{W}^s_O$ will be a n_s-dimensional manifold in Ω_-, an unstable manifold for the map S^{-1}, dense on Ω_-. Likewise we can consider the point $iO \in \Omega_-$ and perform the same construction by using the unstable manifold $W^u_{iO} = iW^s_O$ of iO. It will have a part of dimension $n_u = n_s$ lying on Ω_- and its dimension will be $n_u + n$. The manifolds W^s_O and W^u_{iO} intersect densely on $\Omega_\pm$ and each intersection point $x \in \Omega_+$ is on a n-dimensional manifold which has one point $\tilde{\imath}x \in \Omega_-$ on Ω_-.

It is clear that the densely defined map $\tilde{\imath}$ of $\Omega_+ \longleftrightarrow \Omega_-$ commutes with S and it can be extended by continuity to a map of Ω_+ to Ω_-. If $\mathcal{P}_+$ is a Markov partition of Ω_+ then $\tilde{\imath}\mathcal{P}_+ = \mathcal{P}_-$ is a Markov partition of Ω_-. This is just another way of looking at the construction of the map $\tilde{\imath}$, hence of i^*, see (3.2).

This also shows that we can establish a natural correspondence $\sigma \longleftrightarrow \tilde{\sigma}$ between labels of elements of $\mathcal{P}$ such that $Q_{\tilde{\sigma}} = \tilde{\imath}Q_\sigma$. Note that if $Q_\sigma \in \mathcal{P}_\pm$ then $Q_{\tilde{\sigma}} \in \mathcal{P}_\mp$.

Note that the map i^* has a very simple and natural symbolic dynamics interpretation. Given an allowed sequence $\underline{\sigma} = \{\sigma_j\}$ we set $\underline{\tilde{\sigma}} = \{\tilde{\sigma}_{-j}\}$; since $T_{\sigma,\sigma'} = 1$ means $SQ_\sigma \cap \mathrm{int}(Q_{\sigma'}) \neq \emptyset$, we deduce that it means also $i\,SQ_\sigma \cap i\,\mathrm{int}(Q_{\sigma'}) \neq \emptyset$, hence $S^{-1}iQ_\sigma \cap i\,\mathrm{int}(Q_{\sigma'}) \neq \emptyset$. So that $i\,\mathrm{int}(Q_\sigma) \cap S\,i\,Q_{\sigma'} \neq \emptyset$, hence $i^*Q_\sigma \cap S\,i^*\,\mathrm{int}(Q_{\sigma'}) \neq \emptyset$:

$$T_{\sigma,\sigma'} = 1 \longleftrightarrow Q_{\tilde{\sigma}} \cap S\mathrm{int}(Q_{\tilde{\sigma}'}) \neq \emptyset, \tag{5.1}$$

i.e. $T_{\sigma,\sigma'} = 1$ is equivalent to $T_{\tilde{\sigma}',\tilde{\sigma}} = 1$.

This means that if $\underline{\sigma} = \{\sigma_j\}$ is an allowed sequence of symbols for a point x on the pole Ω_+ (*i.e.* it is the *history* on $\mathcal{P}$ of a point $x \in \Omega_+$ in the sense that $S^j x \in Q_{\sigma_j}$ for all j) then *also* $\underline{\tilde{\sigma}} = \{\tilde{\sigma}_{-j}\}$ is an allowed sequence and i^*x is the (unique) point on the pole Ω_- that has $\{\tilde{\sigma}_{-j}\}$ as the history under S.

6. Markov Partitions, Coarse Graining and Trajectory Segments. Extended Liouville Measure

a) We first discuss the notion of coarse graining, making precise some ideas that were advanced in [G1]. We show that Anosov systems with Axiom A attracting sets admit, in spite of the chaoticity of the motions that they describe, a rather natural decomposition of phase space into cells so that the time evolution can be naturally represented as a *cells permutation* and the SRB distribution can be naturally interpreted as the distribution that gives *equal weight to each of the cells*.

This may look surprising and contradictory with the property of hyperbolicity and chaoticity of the system. Therefore it is a particularly interesting (rather elementary) feature of the SRB distribution which makes it even more analogous to the microcanonical distribution in equilibrium.

Imagine that $\mathcal{P}$ is a Markov partition for a transitive Axiom A attracting set. We use it to set up a symbolic dynamic description of the attractor.

Let T be large and $\mathcal{P}_T = \vee_{j=-\frac{T}{2}}^{\frac{T}{2}} S^j\mathcal{P}$. Then it is well known, [S1,R1] (see also [G2]), that we can represent the SRB distribution as a limit of probability distributions obtained by assigning to the elements $Q \in \mathcal{P}_T$ a weight:

$$\Lambda_{e,T}^{-1}(x_Q), \tag{6.1}$$

where $\Lambda_{e,T}(x)$ is the expansion rate (*i.e.* the modulus of the jacobian determinant) of the map S^T as a map of the unstable manifold of $S^{-T/2}x$ to that of $S^{T/2}x$, and x_Q is a (suitable, see [GC2, G3]) point in Q.

Then we can imagine to partition each $Q \in \mathcal{P}_T$ into boxes so that the number of boxes, that we call *cells*, in Q is proportional to (6.1). In this way we find a representation for the SRB distribution in which each cell of phase space has *the same weight*. The SRB distribution thus appears as the uniform distribution on the attracting set (thus partitioned) and the time evolution can be rather faithfully represented simply as a permutation of the cells, in spite of the hyperbolicity.

This also shows that one has to be careful in saying that "it is obvious that the SRB distribution is obtained by attributing the weights (6.1) to points on the attractor", sometimes erroneously called the "trajectory segment method": this is right only if the points are identified with the cells of a Markov partition $\mathcal{P}_T$, refinement of a fixed Markov partition. This means that (6.1) is correct only if a suitable coarse graining of the phase space is made, and incorrect otherwise.

If the points are chosen differently then the weight to give to each may well be *very different*, as in the latter case in which it is *equal* for all cells, no matter what the value of $\Lambda_{e,T}^{-1}$ is in each of them. And in some sense the latter representation is the most natural one, and it realizes in general the Boltzmann idea that all points in phase space are equivalent and the dynamics is just a one cycle permutation of the cells, [G2].

One can say that if the system admits an Axiom A attracting set then it is possible to define a coarse graining of phase space such that the dynamics is *eventually* just a one cycle cell permutation (*even though* the evolution may be non volume preserving): the SRB distribution appears then as the *uniform distribution* on the relevant phase space part (*i.e.* the attracting set). In other words the chaotic hypothesis can be regarded as a natural version of the original viewpoint of Boltzmann, [G1], on time evolution and ergodicity.

In general the attracting set will support *two* stationary distributions: one, coinciding with μ_+, which describes the statistics of the data that are chosen randomly *on the attracting set* with a distribution proportional to the area elements of the set itself, *and a second one* describing the statistics of the same data evolving backward in time. The latter statistics can be denoted μ_+^* and it is *different* from the statistics of the data that are chosen randomly with distribution proportional to the full phase space volume (the latter is in fact concentrated "elsewhere", on the set obtained from the attracting set by the global time reversal).

A representation of the above dynamical properties in terms of "coarse grain" cells is also easy to set up.

We can imagine to partition *each* $Q \in \mathcal{P}_T$ into boxes of type + and boxes of type $-$ so that the number of boxes of type + in Q is proportional to $\Lambda_{e,T}^{-1}(x_Q)$ and the number of boxes of type $-$ in Q is proportional to $\Lambda_{c,T}(x_Q)$, where this is the modulus of the jacobian determinant of S^T restricted to part of the stable manifold of S on the attracting set, regarded (locally) as a submanifold of the attracting set (that we are supposing to be a manifold).

One can define the forward evolution on the attracting set as a one cycle permutation of the + cells and the backward evolution as a one cycle permutation of the $-$ cells. The local time reversal i^* can now be represented as a one to one transformation of the + cells into the $-$ cells.

The "wandering" points could also be represented as cells of a third type, but they are not interesting for the description of the statistical properties of the motions on the attracting set. The representation of the dynamics in terms of two "compenetrated" sets of coarse grain cells gives a clear idea of how it is possible that the forward and backward evolutions have different statistics related by the same time reversal operation i^* that is the basis of the proof of the fluctuation theorem.

b) Consider an Axiom C system. Then we can define a local time reversal i^* on the future pole. This means that a fluctuation theorem holds for the statistics of the Liouville distribution μ_0 on $\mathcal{C}$. The formulation of the fluctuation theorem is unchanged provided one defines the entropy production rate as the contraction of the surface area on the attracting set. This is to be expected to be a rather difficult quantity to evaluate in concrete cases because we cannot expect to have a precise knowledge of the geometric structure of the attracting set. In this respect one can remark that the system may have other properties that nevertheless allow us to establish a relationship between the contraction rate of the Liouville measure (directly accessible from the measurement of the divergence of the equations of motion) and the contraction rate of the surface measure *on the attracting* *set*: an interesting instance of this has been found in [BGG], Sect. 6, *(ii)*, where the extra property used was the *pairing rule* that held in that case, (see [ECM1, DM]).

In general on the pole Ω_+ one can define a probability distribution μ_0^* that is the natural extension of the Liouville distribution in the equilibrium case and for which the fluctuation theorem holds in the same form that it has in the Anosov case. It is the probability distribution that is defined in the symbolic dynamics representation by the *Gibbs distribution*, [D, LR, R2], with *non translationally invariant potential* given by the formal energy function:

$$H(\underline{\sigma}) = \sum_{k=-\infty}^{-1} h_-(\vartheta^k \underline{\sigma}) + \sum_{k=0}^{\infty} h_+(\vartheta^k \underline{\sigma}), \tag{6.2}$$

where $\underline{\sigma}$ is the symbolic sequence corresponding to a point x on Ω_+ evaluated on a Markov partition $\mathcal{P}$, see Sect. 5; ϑ is the shift of the sequence $\underline{\sigma}$; and we have set:

$$h_-(\underline{\sigma}) = -\log \Lambda_s^*(X(\underline{\sigma})), \quad h_+(\underline{\sigma}) = \log \Lambda_u(X(\underline{\sigma})), \tag{6.3}$$

and Λ_s^*, Λ_u are the jacobian determinants of the map S *restricted* to the intersections of the stable or, respectively, unstable manifolds with Ω_+.

In the Anosov case (6.2) defines a distribution equivalent to the ordinary Liouville distribution, see [G3, G2]. In the Axiom C case it defines a distribution on Ω_+ which is absolutely continuous with respect to the surface area on Ω_+ when the poles are smooth manifolds (because in such a case the system (Ω_+, S) is a Anosov system). But if the pole Ω_+ is just an Axiom A attracting set which is not a smooth manifold then Λ_s^* is not properly defined as a jacobian determinant (because the intersection $\breve{W}_x^s = W_x^s \cap \Omega_+$ is not a manifold). Nevertheless it *can be defined* by using i^* via:

$$\Lambda_s^*(x) = \Lambda_u^{-1}(i^* x), \tag{6.4}$$

and this is our proposal for a natural extension of the definition of the Liouville measure on the attracting basic set Ω_+. It is a distribution that may have several further properties that it seems worth investigating.

c) Finally, with reference to Sect. 1 above, we note that in a system like the one studied in [BGG] it is possible that while a forcing parameter grows the Lyapunov spectrum changes nature because some exponents initially positive continuously evolve into negative ones as the forcing increases. Everytime one "positive" exponent "becomes" negative the dimension of the future pole diminishes (usually by 2 units when the pairing rule is verified, see [BGG]). At this "bifurcation" the future pole splits into two basic sets, one will be the new pole and the other will be its i^* image. Of course the above analysis implies that there will be a new local time reversal i^{**} on the new future pole. The i^* image of the future pole, however, will *not* be the past pole: one can easily see that the latter is more stable than the i–image of the future pole: hence the past pole will be the full time reversal of the future pole, no matter how many intermediate bifurcations took place.

The picture in terms of diagrams, see [Sm] p. 754, is quite suggestive and is that after $n = 0, 1, \ldots$ "positive" Lyapunov exponents have "become" negative the diagram of the system consists of 2^n points totally ordered starting from the past pole and going straight down to the future pole. During the evolution of the bifurcations $n + 1$ time reversals are defined $i_0^* \equiv i, i_1^*, \ldots, i_n^*$ and the $k^{\rm th}$ time reversal i_k^* leaves invariant the set of nodes in the diagram with labels $1, 2, \ldots, 2^{n-k}$, $k = 0, \ldots$.

Acknowledgement. We are indebted to N. Chernov, E.G.D. Cohen, P. Garrido, G . Gentile, J.L. Lebowitz for stimulating discussions and comments. This work is part of the research program of the European Network on: "Stability and Universality in Classical Mechanics", # ERBCHRXCT940460, and it has been partially supported also by CNR-GNFM and Rutgers University.

References

[AA] Arnold, V., Avez, A.: *Ergodic problems of classical mechanics*. New York: Benjamin, 1966

[BGG] Bonetto, F., Gallavotti, G., Garrido, P.: *Chaotic principle: an experimental test.* Roma, Preprint, (1996)

[CG1] Gallavotti, G., Cohen, E.G.D.: Dynamical ensembles in nonequilibrium statistical mechanics. Phys. Rev. Let. **74**, 2694–2697 (1995)

[CG2] Gallavotti, G., Cohen, E.G.D.: Dynamical ensembles in stationary states. In print in J. Stat. Phys. (1995)

[D] Dobrushin, R.L.: Gibbs random fields for lattice systems with pair interactions. Functional Anal. and Appl. vol. **2**, 31–43, 1968

[DM] Dettman, Morriss, G.: Preprint 1995

[ECM1] Evans, D.J., Cohen, E.G.D., Morriss, G.P.: Viscosity of a simple fluid from its maximal Lyapunov exponents. Phys. Rev. **42A**, 5990–5997 (1990)

[ER] Eckmann, J.P., Ruelle, D.: Ergodic theory of strange attractors. Rev. Mod. Phys. **57**, 617–656 (1985)

[G1] Gallavotti, G.: Ergodicity, ensembles, irreversibility in Boltzmann and beyond. J. Stat. Phys. **78**, 1571–1589 (1995)

[G2] Gallavotti, G.: Topics in chaotic dynamics. Lectures at the Granada School, ed. Garrido–Marro, Lecture Notes in Physics **448**, 1995

[G3] Gallavotti, G.: Reversible Anosov diffeomorphisms and large deviations. Mathematical Phys. Electr. J. **1**, (1) (1995)

[G4] Gallavotti, G.: Chaotic hypothesis: Onsager reciprocity and fluctuation–dissipation theorem. J. Stat. Phys. **84**, 899–926 (1996)

[LR] Lanford, O., Ruelle, D.: Observables at infinity and states with short range correlations. Commun. Math. Phys. **9**, 327–338 (1968)

[R1] Ruelle, D.: Chaotic motions and strange attractors. Lezioni Lincee, notes by S. Isola, Accademia Nazionale dei Lincei, Cambridge: Cambridge University Press, 1989; see also: Ruelle, D.: Measures describing a turbulent flow. Annals of the New York Academy of Sciences, **357**, 1–9 (1980). For more technical expositions see Ruelle, D.: Ergodic theory of differentiable dynamical systems. Pub. Math. de l' IHES **50**, 275–306 (1980)

[R2] Ruelle, D.: A measure associated with Axiom A attractors. Am. J. Math. **98**, 619–654 (1976)

[R3] Ruelle, D.: *Elements of differentiable dynamics and bifurcation theory.* New York: Academic Press, 1989

[S1] Sinai, Y.G.: Gibbs measures in ergodic theory. Russ. Math. Surv. **27**, 21–69 (1972). Also: *Introduction to ergodic theory*. Princeton: Princeton U. Press, Princeton, 1977

[Sm] Smale, S.: Differentiable dynamical systems. Bull. Am. Math. Soc. **73**, 747–818 (1967)

[UF] Uhlenbeck, G.E., Ford, G.W.: *Lectures in Statistical Mechanics.* Providence, RI: Am. Math. Soc., 1963, pp. 5, 16, 30

Communicated by J.L. Lebowitz