Critical indices in a $d=1$ filled-band Fermi system

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By renormalization-group methods we obtain nonperturbative results about a $d=1$ system of interacting spinless fermions in a periodic potential when the conduction band is filled. Both the strength of the interaction and the amplitude of the periodic potential are assumed to be small. We determine that the large-distance asymptotic behavior of the two-point Schwinger function is anomalous and described by two critical indices, explicitly computed by convergent series, related to the renormalization of the spectral gap and of the discontinuity at the Fermi surface. [S0163-1829(97)03023-3]

I. INTRODUCTION

The study of $d=1$ systems of interacting fermions has attracted interest, not only because they can describe[1] suitable strongly anisotropic compounds but also because they have a very rich structure, and their properties may give, hopefully, hints on the intricacies that might be present in higher-dimensional models: for instance the properties of some $d=2$ compounds which manifest high-T_c superconductivity are explained by assuming that they have some properties typical of one-dimensional systems[2] (''Luttinger liquid behavior'').

The standard Hamiltonian of a system of one-dimensional fermions with spin σ that move in a periodic force $-u\partial_{\mathbf{x}}c(\mathbf{x})$, $c(\mathbf{x}+a)=c(\mathbf{x})$, and interact via a short-range pair potential $\lambda v(\mathbf{x}-\mathbf{y})$ is

$$H(\phi)=T(\phi)+uP(\phi)+\lambda\widetilde{V}(\phi)+\nu\overline{N(\phi)}, \tag{1}$$

$$T(\phi)=\int_{-L/2}^{L/2}d\mathbf{x}\phi^{+}_{\mathbf{x},\sigma}\left[-\frac{\hbar^2\partial^2_{\mathbf{x}}}{2m}-\mu\right]\phi^{-}_{\mathbf{x},\sigma},$$

$$P(\phi)=\int_{-L/2}^{L/2}d\mathbf{x}\phi^{+}_{\mathbf{x},\sigma}c(\mathbf{x})\phi^{-}_{\mathbf{x},\sigma},$$

$$\widetilde{V}(\phi)=-\int_{-L/2}^{L/2}d\mathbf{x}\,d\mathbf{y}v(\mathbf{x}-\mathbf{y})\phi^{+}_{\mathbf{x},\sigma}\phi^{-}_{\mathbf{x},\sigma}\phi^{+}_{\mathbf{y},\sigma'}\phi^{-}_{\mathbf{y},\sigma'},$$

$$\overline{N(\phi)}=\int_{-L/2}^{L/2}d\mathbf{x}\phi^{+}_{\mathbf{x},\sigma}\phi^{-}_{\mathbf{x},\sigma},$$

where $\phi^{+}_{\mathbf{x},\sigma}$, $\phi^{-}_{\mathbf{x},\sigma}$ are creation or annihilation fermionic field operators with spin σ on the Fock space of fermions confined in a box $\Lambda=[-L/2,L/2]$ with periodic-boundary conditions, obeying the anticommutation rule. If $\sigma=0$ we say that the fermions are spinless while if $\sigma=\pm1/2$ they are spinning, and in this case the sum over the spins is understood. We take $L=Na$, where N is an integer. We assume that u and λ are dimensionless and $c(\mathbf{x})=(\hbar^2/2ma^2)\widetilde{c}(\mathbf{x}/a)$, $v(\mathbf{r})=(\hbar^2/2ma^2)\widetilde{v}(\mathbf{r}p_0)$, with $\widetilde{c}(\mathbf{x})$, $\widetilde{v}(\mathbf{r})$ also dimensionless; moreover it is not restrictive to assume $u\geqslant0$ and $\int_0^a c(\mathbf{x})=0$ and we choose the sign of λ assuming that $\hat{v}(0)-\hat{v}(2p_F)\geqslant0$. Finally $\widetilde{c}(\mathbf{x})$ and $\widetilde{v}(\mathbf{r})$ are assumed to be rotationally invariant, i.e., even; u is called the amplitude of the periodic potential, λ is the strength of the interaction, and m is the fermion mass, ν is a counterterm to be fixed to a λ, u-dependent value in order to fix the Fermi momentum (see below). We call $V(\phi)=uP(\phi)+\nu\overline{N(\phi)}+\lambda\widetilde{V}(\phi)$, and we denote by E_0^n (ground-state energy) the minimum value of $\underline{H}$ over the states $|\psi_n\rangle$ with n particles, i.e., such that $\overline{N}|\psi_n\rangle=n|\psi_n\rangle$.

The grand-canonical state at volume Λ, inverse temperature β, and chemical potential $\mu+\nu$ can be obtained by the Schwinger functions (the imaginary-time Green functions[3]):

$$S_n^{L,\beta}(x_1,\sigma_1,\varepsilon_1;\ldots;x_{2n},\sigma_{2n},\varepsilon_{2n})=\frac{\int\{\mathcal{D}\psi e^{-\int dk\psi^{+}_{k,\sigma}[-ik_0-E(\mathbf{k})]\psi^{-}_{k,\sigma}}\}e^{-V(\psi)}\prod_{i=1}^{2n}\psi^{\varepsilon_i}_{x_i,\sigma_i}}{\int\{\mathcal{D}\psi e^{-\int dk\psi^{+}_{k,\sigma}[-ik_0-E(\mathbf{k})]\psi^{-}_{k,\sigma}}\}e^{-V(\psi)}} \tag{2}$$

where $\varepsilon_i=\pm$, $x=(x_0,\mathbf{x})$, $k=(k_0,\mathbf{k})=2\pi[(n_0+2^{-1})/\beta,n_1/L]$, if n_0, n_1 are integers, $\hbar E(\mathbf{k})=(\hbar^2\mathbf{k}^2/2m)-\mu$, $\psi^{\pm}_{x,\sigma}$, $\psi^{\pm}_{k,\sigma}$, are Grassman variables and from now on we denote by $\int\{\mathcal{D}\psi e^{-\int dk\psi^{+}_{k,\sigma}h(k_0,\mathbf{k})^{-1}\psi^{-}_{k,\sigma}}\}$ the fermionic integration, a linear-operator defined over the monomials of Grassman variables $\psi^{\pm}_x$ by the anticommutative Wick rule with propagator $\int dk\,e^{ik(x-y)}h(k_0,\mathbf{k})$ where $\int dk=1/\beta L\Sigma_k$; we set

$$g^{L,\beta}(x-y)=\int dk\,\frac{e^{ik(x-y)}}{-ik_0-E(\mathbf{k})}. \tag{3}$$

In this paper we study the two-point Schwinger function defined as

$$S_2^{L,\beta}(x,\sigma,+;y,\sigma,-)\equiv S^{L,\beta}(x,y). \tag{4}$$

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We call $\lim_{L,\beta\to\infty}S^{L,\beta}(x,y)=S(x,y)$.

If $\lambda=\nu=0$ the n-particle eigenfunctions of H are the antisymmetrized product of the one-particle wave functions, $\phi(\mathbf{k},\mathbf{x},u)$, solving the Schrödinger equation with periodic potential $uc(\mathbf{x})$:

$$\left[\frac{-\hbar^2\partial_{\mathbf{x}}^2}{2m}+uc(\mathbf{x})\right]\phi(\mathbf{k},\mathbf{x},u)=\varepsilon(\mathbf{k},u)\phi(\mathbf{k},\mathbf{x},u).$$

The functions $\phi(\mathbf{k},\mathbf{x},u)$ are called Bloch waves. We define

$$\hat{S}^{L,\beta}(\mathbf{k};k_0)=\frac{1}{L}\int d\mathbf{x}\,d\mathbf{y}\,d(x_0-y_0)e^{-ik_0(x_0-y_0)}\times\phi(\mathbf{k},-\mathbf{x},u)\phi(\mathbf{k},\mathbf{y},u)S_2^{L,\beta}(x,y),$$

$$\overline{S^{L,\beta}}(\mathbf{k};t)=\frac{1}{L}\int d\mathbf{x}\,d\mathbf{y}\,\phi(\mathbf{k},-\mathbf{x},u)\phi(\mathbf{k},\mathbf{y},u)S_2^{L,\beta}(x,y).$$

Many important physical properties can be obtained from the two-point Schwinger function. The occupation number is defined as

$$n_{\mathbf{k}}^{L,\beta}=\overline{S^{L,\beta}}(\mathbf{k};0^-)$$

and $n_{\mathbf{k}}=\lim_{L,\beta\to\infty}n_{\mathbf{k}}^{L,\beta}$. The Fermi momentum $p_F(\lambda,u,\mu+\nu)$ in a $d=1$ fermionic rotation-invariant theory with chemical potential $\mu+\nu$ is defined requiring that $n_{\mathbf{k}}$ is not regular, i.e., $n_{\mathbf{k}}$ or some of its derivatives are singular at $\mathbf{k}=\pm p_F(\lambda,u,\mu+\nu)$. If there are more than two points where the occupation number is not regular, the Fermi momentum is chosen such that it can be continuously parametrized by λ and it reduces for $\lambda=0$ to the Fermi momentum of the free $\lambda=0$ theory. Since the early works on the theory of Fermi systems,[4] it has been realized that it is more natural to study the properties of weakly interacting fermionic systems when λ is varied at fixed Fermi momentum rather than at fixed-chemical potential. Therefore, we fix ν so that the Fermi momentum in the interacting $\lambda\neq0$ system, with chemical potential $\mu+\nu$, is equal to the Fermi momentum of the free $\lambda=0$ system with chemical potential equal to μ, i.e., $p_F(\lambda,u,\mu+\nu)=p_F(0,u,\mu)\equiv p_F$. The two points $\pm p_F$ are called Fermi surface. The Fermi momentum when L,β is finite can be chosen as $(2\pi/L)(n_F^L+1/2)$, if n_F^L is an integer such that $(2\pi/L)(n_F^L+1/2)\to_{L\to\infty}p_F$. The discontinuity at the Fermi surface is defined as

$$Z^{-1}=n_{p_F^-}-n_{p_F^+}. \tag{5}$$

The spectral gap

$$\bar{\Delta}=E_0^{n+1}+E_0^{n-1}-2E_0^n \tag{6}$$

can be computed from the imaginary poles in k_0 of $\hat{S}(\mathbf{k};k_0)$; if $i\hbar k_{0,\alpha}$, $-i\hbar k_{0,\beta}$, $k_{0,\alpha}$, $k_{0,\beta}>0$ are such poles, then it is well known that $\Delta=\min_\alpha(k_\alpha)+\min_\beta(k_\beta)$.

It is easy to check by an explicit computation that the Schwinger functions for the free $\lambda=\nu=0$ particle system with Hamiltonian $T+uP$ are given by the Wick's rule in term $S_0^{L,\beta}(x,y)$:

$$S_0^{L,\beta}(x,y)=\int dk\,\frac{\phi(\mathbf{k},\mathbf{x},u)\phi(\mathbf{k},-\mathbf{y},u)e^{ik_0(x_0-y_0)}}{-ik_0-[\varepsilon(\mathbf{k},u)-\mu]\hbar^{-1}}. \tag{7}$$

The occupation number is given by $n_{\mathbf{k}}=\chi[\varepsilon(\mathbf{k},u)-\mu\leqslant0]$ where χ(condition) is 1 if the condition is verified and 0 otherwise. The Fermi momentum p_F is then defined by the condition $\varepsilon(p_F,u)=\mu$. The discontinuity at the Fermi surface is $Z^{-1}=1$. The spectral gap is equal to $\varepsilon[(q\pi/a)^+,u]-\varepsilon[(q\pi/a)^-,u]$ where $p_F=q\pi/a$, q is an integer, and it is 0 for all the other values of p_F. It is convenient to define the adimensional spectral gap, if $p_F=q\pi/a$, as $\Delta_q=(2ma^2/\hbar^2)[\varepsilon\{(q\pi/a)^+,u\}-\varepsilon\{(q\pi/a)^-,u\}]$. For small u we have $\Delta_1\equiv\Delta=c_1u+O(u^2)$ where c_1, the first Fourier coefficient of $\tilde{c}(\mathbf{x})$, is assumed from now on equal to 1. If $p_F=q\pi/a$ the system is called filled-band Fermi system. The asymptotic behavior for large values of $|x-y|=a^{-1}\sqrt{(\mathbf{x}-\mathbf{y})^2+v_0^2(x_0-y_0)^2}$ of the two-point Schwinger function depends critically on the value of the Fermi momentum:[5,6] (1) if $p_F\neq q\pi/a$ the two-point Schwinger function decays as $f(\mathbf{x},\mathbf{y})/|x-y|$, where $f(\mathbf{x},\mathbf{y})$ is a bounded function equal to $\sin[p_F(\mathbf{x}-\mathbf{y})]$ if $u=0$, i.e., it behaves for large distances as in the $u=0$ case; (2) if $p_F=q\pi/a$ for any $N>1$ one can find constants C_N, C such that

$$|S_0(x,y)|\leqslant\frac{C_N\Delta_q a^{-1}}{1+\Delta_q^N|x-y|^N} \tag{8}$$

for $|x-y|>\Delta_q^{-1}$, while

$$|S_0(x,y)|\leqslant\frac{Ca^{-1}}{|x-y|}$$

for $1\leqslant|x-y|\leqslant\Delta_q^{-1}$, if $\Delta_q<1$.

In the filled-band case the large-distance behavior of the two-point Schwinger function is then discriminated by an intrinsic length, which is $O(u)$ if the strength of the periodic potential is small.

In the interacting $\lambda\neq0$ case, the computation of $S(x,y)$ is not an easy task. In general it is not directly studied for the Hamiltonian Eq. (1) but for different ones describing two kinds of interacting fermions with a linear dispersion relation (g-ology models).[1,7,8] The relation between the g-ology model and the model Eq. (1) is quite clear in a renormalization group approach and it will be discussed in Sec. III. The simplest of these models is the Luttinger model;[9] it was proved[10] that its Hamiltonian can be diagonalized in terms of suitable bosonic operators and that the asymptotic behavior of the two-point Schwinger function for large distances is anomalous as $S(x,y)=[g_0(x-y)/|x-y|^\eta]+\lambda[A(x,y)/|x-y|^{1+\eta}]$, with $A(x,y)$ bounded by a constant and $g_0(x-y)$ is the free Luttinger-model Schwinger function; moreover $\Delta=0$, $Z^{-1}=0$ and $n_{p_F-\varepsilon}-n_{p_F+\varepsilon}=O(\varepsilon^\eta)$. Also the explicit expression for the n-particle Schwinger functions can be obtained.[11,12] The importance of the Luttinger model is that there is a wide class of models whose properties are similar to the Luttinger model ones; such models are called Luttinger liquids.[13] Many other g-ology models were introduced in the literature; the Mattis model,[14] the massive Luttinger model,[7] the Luther-Emery model,[15] and the Umklapp model.[16] Unfortunately, they are not exactly soluble, as their Schwinger functions are not known. There are many results on these models but they are obtained by the so-called bosonization method whose validity is not really clear.[1]

In recent times, the techniques developed in constructive quantum-field theory[17] for producing examples of nontrivial quantum-field models were applied to study the model Eq. (1).[18] The aim was to obtain the Schwinger functions nonperturbatively proving the summability of the series expressing them. The existence of the two-point Schwinger function is proved[19–21] for the model Eq. (1) in the spinless $u=0$ case by showing the uniform convergence in L,β (at small coupling) of the series expressing it. Its behavior for large distances is anomalous (Luttinger-liquid behavior). The same properties hold[5] for the $u\neq 0$ spinless case, if $p_F\neq q\pi/a$ (not-filled band case), and for the $u\neq 0$ spinning case, if $p_F\neq q\pi/a$, $q\pi/2a$ (not-filled nor half-filled band case), and the interaction is repulsive. In the spinning case, when the interaction is attractive or when the band is half-filled, there are no available rigorous results; despite this many interesting conjectures about this case are present in the literature (see below). In this paper, we study nonperturbatively the Schwinger function of the Hamiltonian Eq. (1) in the spinless case for $p_F=\pi/a$ (similar results of course hold for $p_F=q\pi/a$) i.e., for a value of the Fermi momentum corresponding to the filled-band case. Despite the fact that we do not find in this case a Luttinger-liquid behavior, which is expected only in partially filled-band models, our model shows an anomalous behavior which reduces to a Luttinger-liquid behavior as $u\to 0$.

II. MAIN RESULTS

We consider both λ, u to be small; the case of small λ and large u (in particular greater than $C|\lambda|$, if C is a proper constant) can be treated by considering as Grassmanian integration the one defined by $S_0^{L,\beta}(x,y)$ Eq. (7) instead of $g^{L,\beta}(x-y)$ Eq. (3), i.e., by considering the periodic potential term in the Hamiltonian not as a perturbation but as a part of the fermionic integration; the large-distance behavior of the two-point Schwinger function is substantially identical to the free one in Eq. (7), so that this ''trivial'' case is not discussed here.[6]

In the spinless case we prove that there exist an $0<\varepsilon\ll 1$ and a $\nu=O(\lambda^2)$ (Ref. 22) such that, for $|u|$, $|\lambda|\leqslant\varepsilon$ the following properties hold.

(1) *Decay of the two-point Schwinger function*. $S(x,y)$ is such that, for $|x-y|>\hat u(p_F)^{-1}$

$$|S(x,y)|\leqslant \hat Z(p_F)^{-1}\,\frac{C_N\hat u(p_F)a^{-1}}{1+(\hat u[p_F]|x-y|)^N} \tag{9}$$

for any $N>1$, where C_N is a suitable constant, while for $1\leqslant|x-y|\leqslant\hat u(p_F)^{-1}$:

$$|S(x,y)|\leqslant C|x-y|^{-1-\eta_3}, \tag{10}$$

where C is a constant and

$$\hat u(p_F)=u^{1+\eta_2},\quad \hat Z(p_F)^{-1}=u^{\eta_1} \tag{11}$$

with $\eta_1=\beta_3\lambda^2+O(\lambda^3)$ and $\eta_2=\beta_1\lambda+O(\lambda^2)$, $\beta_1,\beta_3>0$, $\eta_3=\eta_1(1+\eta_2)^{-1}$. This means that, like in the $\lambda=0$ case [see Eq. (8)], one can distinguish two regions in the large distance behavior of the two-point Schwinger function, discriminated by an intrinsic length produced by the periodic potential, which now is changed by the interaction from $O(u)$ to $O[\hat u(p_F)]$. In the first region, again, the two-point Schwinger function decays faster than any power, with the difference that in the bound the decay rate $O(u)$ is replaced by $\hat u(p_F)$ and there is an extra factor $\hat Z(p_F)^{-1}$. In the second region, it is bounded by a power-law bound, but the exponent is no longer 1 but $1+\eta_3$. Note the remarkable fact that, contrary to what happens in the Luttinger liquids, the large-distances asymptotic decay of the two-point Schwinger function is described in terms of two critical indices, not by one. Note also that in the limit $\lambda\to 0$ or $u\to 0$ one should recover the expected behavior.

(2) *Anomalous occupation number discontinuity*. There are two positive constants c_1 and c_2 such that Z^{-1}, Eq. (5) is

$$c_1\hat Z(p_F)^{-1}\leqslant Z^{-1}\leqslant c_2\hat Z(p_F)^{-1}. \tag{12}$$

If $u\ll e^{-\kappa_1\lambda^{-2}}$, where κ_1 is a constant, the interaction decreases dramatically in the presence of the discontinuity at the Fermi surface, i.e., $Z^{-1}\ll 1$; in fact Z^{-1} is vanishing as $u\to 0$ as $O(u^{\eta_1})$, in agreement with the Luttinger behavior of the $u=0$ case in which $Z^{-1}=0$.

(3) *Anomalous spectral gap*. It is possible to show that

$$\overline{\Delta}\geqslant\frac{1}{2}\,\hbar v_0 a^{-1}\hat u(p_F). \tag{13}$$

There is a nonvanishing spectral gap also in the presence of an interaction but, at least if the interaction is attractive ($\lambda<0$) and $u\ll e^{-\kappa_1|\lambda|}$, it is strongly renormalized by the interaction as the ratio between the bare gap and the dressed gap is $\ll 1$ for small u and vanishing as $u\to 0$.

(4) *Interacting Bloch waves*. The two-point Schwinger function can be written as $S(x,y)=S_A(x,y)+\tilde\epsilon S_B(x,y)$ with

$$S_A(x,y)=\int\frac{1}{\hat Z(k)}\,\frac{\phi[\mathbf{k},\mathbf{x},\hat u(k)]\phi[\mathbf{k},-\mathbf{y},\hat u(k)]e^{ik_0(x_0-y_0)}}{-ik_0-\{\varepsilon[k,\hat u(k)]-\mu\}\hbar^{-1}} \tag{14}$$

and $\tilde\epsilon=\max\{|\lambda|,u,\hat u(p_F)\}$, $\phi(\mathbf{k},\mathbf{x},u)$, $\varepsilon(\mathbf{k},u)$ are the Bloch wave and its correspondent eigenvalue, and $\hat u(k)$ and $\hat Z^{-1}(k)$ are two regular functions such that $|\hat u(k)-u|=O(u\lambda)$, $|\hat Z^{-1}(k)-1|=O(\lambda)$ for $|k|>\pi/2a$, and $\hat u(p_F)$, $\hat Z^{-1}(p_F)$ are given by Eq. (11).

The Schwinger function can be written then as the sum of two terms with $\lim_{\lambda,u\to 0}S_A=g$ and $\lim_{\lambda,u\to 0}\tilde\epsilon S_B=0$. S_A is formally similar to the $\lambda=0$ Schwinger function S_0, but the amplitude of the periodic potential u and the wave-function normalization are replaced by $\hat u(k)$ and $\hat Z(k)$. If S_A were the ''dominant'' part of the Schwinger function, at least for large distances (what we are not able to prove), one could interpret this fact saying that the interacting one-particle wave functions for $\mathbf{k}$ near p_F are approximately, i.e., neglecting corrections, $\phi[\mathbf{k},\mathbf{x},\hat u(k)]/\sqrt{\hat Z(k)}$, i.e., interacting Bloch waves. This extra momentum dependence is natural as we expect that the interaction changes the one-particle wave functions mainly for momenta near the Fermi surface. One can expect then, that the spectral gap, which in the noninteracting $\lambda=0$ case is $O(u)$, is deeply renormalized by the interaction between electrons becoming $O(u^{1+\eta_2})$, and becoming much larger or much smaller, if $u \ll e^{-\kappa_1|\lambda|}$, depending on the attractive or repulsive nature of the interaction.

Unambiguous experimental observations of the above properties are made difficult by two facts. The first is that the model is one dimensional, but all possible candidates are quasi-one-dimensional as they consist of a parallel arrangement of conducting chains and it is not obvious that one can neglect the inter-chain hopping. Moreover, our results are valid if one neglects the spin, but the presence of the spin can be relevant.

However, like in the case of the Luttinger model, one can identify a class of models whose behavior is very close to the one we found for the model Eq. (1) in the spinless filled-band case. For some of these models this ''equivalence'' has a clear sense, in the sense that the dominant part of their (infrared) β function is identical to the β function of our model (see the last section). This is the case of the Yukawa$_2$ model or of the massive Luttinger model with Hamiltonian

$$\int d\mathbf{x} \sum_{\boldsymbol{\omega}=\pm 1} \psi^+_{\mathbf{x},\boldsymbol{\omega}}(\boldsymbol{\omega}\partial_{\mathbf{x}}-\mu)\psi_{\mathbf{x},\boldsymbol{\omega}}+\Delta\int d\mathbf{x}\sum_{\boldsymbol{\omega}=\pm 1}\psi^+_{\mathbf{x},\boldsymbol{\omega}}\psi_{\mathbf{x},-\boldsymbol{\omega}} -\lambda\int d\mathbf{x}\int d\mathbf{y}\,v(\mathbf{x}-\mathbf{y})\psi^+_{\mathbf{x},1}\psi^+_{\mathbf{y},-1}\psi_{\mathbf{y},-1}\psi_{\mathbf{x},1}. \tag{15}$$

It will be clear from the following sections that all the above statements are valid, up to trivial modification, for the above model. Then our result (3) confirms the belief (proved by heuristic arguments) that the interacting gap in the massive Luttinger model is[7,23] $\Delta^{1+\eta}$, $\eta=O(\lambda)$. Note that the physical statements 1, 2, and 4 are, as far as we know, not present in the literature even heuristically.

On the basis of bosonization methods, many other models are believed to belong to the universality class of our model, or of the massive Luttinger model. We mention (1) the Umklapp model,[16] describing spinning fermions interacting in the half-filled-band case, when Umklapp scattering is relevant, (2) the Luther-Emery model,[15] describing spinning fermions with an attractive interaction; and (3) the Sine-Gordon model with a suitable identification of the couplings.[24]

In fact,[1,8] the Hamiltonian of the Umklapp model is reduced, by a bosonization transformation (whose validity is however not clear, see below), to the Hamiltonian of the massive Luttinger model Eq. (15), with the identification $\lambda=g_2$, $\Delta=g_3/2\pi\alpha$, if g_2,g_3 are parameters appearing in the Hamiltonian (the backward and umklapp scattering coupling) and α is a parameter appearing in the bosonization transformation. In the same way the Hamiltonian of the Luther-Emery model is reduced to the Hamiltonian of the massive Luttinger model with the identification $\lambda=g_2$, $\Delta=g_1/2\pi\alpha$, if g_1 is the forward scattering coupling. It is, of course, crucial to have rigorous results about the physical properties of these models, as many physical situations are in fact interpreted in terms of the models 1–3.[1,8] In particular, by accepting the reduction of these models to the massive Luttinger model, the relations Eqs. (9)–(14) are indeed verified in these models.

The definition of a perturbative expansion for the Schwinger function and the mathematical proof of its convergence are in another paper;[6] here, we discuss how to derive from such proof the physical statements 1–4. Moreover, we try to give an intuitive idea on the methods we used, as we think they are of interest for physicists. There are many advantages in using these techniques at least in this case, if compared with the techniques usually adopted in literature.

(1) We do not need to know that the fermionic dispersion-relation is linear; this is required in the bosonization, or in the standard-multiplicative renormalization-group approach. The description in terms of two kinds of particles emerges naturally in our approach and the different kinds of couplings in the g-ology model correspond to the relevant part of the interaction.

(2) Although results similar to ours could perhaps be found by bosonization methods in the massive Luttinger model, we stress that the validity of this technique, with the remarkable exception of its application in the Luttinger model,[10] is not justified mathematically.[1,25] As a consequence, the physical quantities computed by the bosonization also depend on a parameter α which does not appear in the Hamiltonian and whose physical meaning is not clear. This makes problematic the comparison of the above quantities with the experimental data.

(3) On the other hand, the convergence of the series expressing the physical quantities allows us to compute them with a fixed and known precision, which seems crucial for a comparison with physical experiments. In the standard multiplicative renormalization-group approach, the calculations are confined to the lowest orders and the problem of the convergence of the series is not considered.

III. THE PERTURBATIVE EXPANSION

We start by integrating the denominator of Eq. (2), the partition function $\mathcal{N}$. We set for simplicity $a=\hbar=2m=1$. This implies that $p_F=\pi$, $v_0=2\pi$.

We perform a decomposition of the propagator:

$$\frac{1}{-ik_0-\mathbf{k}^2+\pi^2}=\frac{1-C_0^{-1}(k)}{-ik_0-\mathbf{k}^2+\pi^2}+\frac{C_0^{-1}(k)}{-ik_0-\mathbf{k}^2+\pi^2}, \tag{16}$$

where $C_0^{-1}(t)$ is a $C^\infty(R^+)$ function which is 0 for $t<p_F/2$ and it is equal to 1 for $t>\gamma(p_F/2)$, if $\gamma>1$ is a scaling parameter; moreover $C_0^{-1}(k)\equiv C_0^{-1}[\sqrt{k_0^2+(|\mathbf{k}|-p_F)^2}]$, $E(k)=\mathbf{k}^2-p_F^2$, and we call the two addends, respectively, $g^{(\leq 0)}(k)$ and $g^{(>0)}(k)$. Equation (16) allows us to represent $\psi^\pm_{k,\sigma}$ as the sum of two independent Grassmanian variables, $\psi^{\pm(>0)}_{k,\sigma}$, $\psi^{\pm(\leq 0)}_{k,\sigma}$ with Grassmanian integrations $\{\mathcal{D}\psi^{(>0)}e^{-\int dk\,\psi^{+(>0)}_{k,\sigma}g^{(>0)}(k)^{-1}\psi^{-(>0)}_{k,\sigma}}\}$ and $\{\mathcal{D}\psi^{(\leq 0)}e^{-\int dk\,\psi^{+(\leq 0)}_{k,\sigma}g^{(\leq 0)}(k)^{-1}\psi^{-(\leq 0)}_{k,\sigma}}\}$. The integration on $\psi^{(>0)}$ of $\mathcal{N}$

$$\int \{\mathcal{D}\psi^{(>0)}e^{-\int dk\,\psi^{+(>0)}_{k,\sigma}g^{(>0)}(k)^{-1}\psi^{-(>0)}_{k,\sigma}}\}e^{-V(\psi^{(\leq 0)}+\psi^{(>0)})} \equiv e^{-V^0(\psi^{(\leq 0)})} \tag{17}$$

gives[21,5]

$$V^0(\psi)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\int\prod_{i=1}^{2m}dk_i W_{m,n}(k_1,\ldots,k_{2m};z)\times\prod_{i=1}^{m}\psi^+_{k_i,\sigma_i}\prod_{i=m+1}^{2m}\psi^-_{k_i,\sigma_i}\times\delta\left(\sum_{i=1}^{m}k_i-\sum_{i=m+1}^{2m}k_i+2n\pi\right),\tag{18}$$

where $2n\pi$ is a spatial vector and the kernels $W_{m,n}(k_1,\ldots,k_m;z)$ are C^∞ bounded functions such that $W_{m,n}=W_{m,-n}$ and $|W_{m,n}|\le C^m z^{\max(2,m-1)}$ if $z=\max(\lambda,u,\nu)$.

After the integration of the ultraviolet field components, i.e., of $\psi^{(>0)}$, the problem is reduced to an essentially identical one but with a purely "infrared" propagator $g^{(\le 0)}$ and a new potential with terms of an arbitrary degree in the fields. In order to perform the integration $\int\{\mathcal{D}\psi^{(\le 0)}e^{-\int dk\psi^{+(\le 0)}_{k,\sigma}g^{(\le 0)}(k)^{-1}\psi^{-(\le 0)}_{k,\sigma}}\}e^{-V^0(\psi^{\le 0})}$, the propagator $g^{(\le 0)}(k)$ is decomposed as[26]

$$g^{(\le 0)}(k)=\frac{1-C_0^{-1}(k)}{-ik_0-E(\mathbf{k})}=\sum_{h=-\infty}^{0}\frac{f_h(k)}{-ik_0-E(\mathbf{k})}\equiv\sum_{h=-\infty}^{0}g^{(h)}(k)\tag{19}$$

with $f(t)=C_0^{-1}(\gamma t)-C_0^{-1}(t)$, and $f_h(k)\equiv f[\gamma^{-h}\sqrt{k_0^2+(|\mathbf{k}|-p_F)^2}]$. This allows us to write $\psi^{(\le 0)}_{k,\sigma}=\Sigma_{h=-\infty}^{0}\psi^{(h)}_{k,\sigma}$. The numbers h will be called scales.

Since $g^{(h)}(k)$ does not have good scaling properties (it contains an intrinsic scale length p_F^{-1}) it is convenient to write the propagator $g^{(h)}(k)$ as $g^{(h)}(k)=\Sigma_{\boldsymbol{\omega}=\pm 1}g^{(h)}_{\boldsymbol{\omega}}(k)$ with

$$g^{(h)}_{\boldsymbol{\omega}}(k)=\chi(\boldsymbol{\omega}\mathbf{k})\frac{f_h(k)}{-ik_0-\mathbf{k}^2+p_F^2},\tag{20}$$

where $\chi(\mathbf{k})$ is the step function, i.e., $\chi(\mathbf{k})=0$ if $\mathbf{k}<0$ and $\chi(\mathbf{k})=1$ otherwise. The χ functions select the two quasiparticles with momenta close to $\pm p_F$ (Ref. 17) with propagator $g^{(h)}_{\boldsymbol{\omega}}(k)$; note that $g^{(h)}_1(k)$ and $g^{(h)}_{-1}(k)$ are compact support functions with disjoint supports. If $\boldsymbol{\omega}\mathbf{k}>0$ we will write in the following $\mathbf{k}=\mathbf{k}'+\boldsymbol{\omega}p_F$, where $\mathbf{k}'$ is the momentum measured from the Fermi surface and we shall use the notation $k'=(\mathbf{k}',k_0)$. We can write $g^{(h)}_{\boldsymbol{\omega}}(k'+\boldsymbol{\omega}\pi)=\gamma^{-h}g_{\boldsymbol{\omega}}(\gamma^{-h}k')+\overline{g^{(h)}}(\gamma^{-h}k')$ with $\overline{g^{(h)}}(k')$ regular and weakly dependent on h and

$$g_{\boldsymbol{\omega}}(k')=\frac{f[k_0^2+(\mathbf{k}')^2]}{-ik_0-\boldsymbol{\omega}\pi\mathbf{k}'}.$$

This induces a decomposition of $\psi^{(h)}_{k,\sigma}=\Sigma_{\boldsymbol{\omega}=\pm 1}\psi^{\pm(h)}_{k'+\boldsymbol{\omega}p_F,\boldsymbol{\omega},\sigma}$ and $\psi^{\pm(h)}_{k'+\boldsymbol{\omega}p_F,\boldsymbol{\omega},\sigma}$ to have a distribution which, up to scaling is essentially h independent, i.e., the distribution of $\psi^{h,\pm}_{k'+\boldsymbol{\omega}p_F,\boldsymbol{\omega},\sigma}$ is the same of $\gamma^{-h/2}\psi^{(h),\pm}_{\gamma^{-h}k'+\boldsymbol{\omega}p_F,\boldsymbol{\omega},\sigma}$ up to corrections negligible in the $h\to-\infty$ limit. From the renormalization-group analysis of the model with Hamiltonian Eq. (1) naturally emerges a description in terms of two kinds of quasiparticles labeled by $\boldsymbol{\omega}=\pm 1$ with linear dispersion relation. The Fourier transform of $g^{(h)}_{\boldsymbol{\omega}}(k)$, called $g^{(h)}_{\boldsymbol{\omega}}(x)$, verifies the bound $|g^{(h)}_{\boldsymbol{\omega}}(x)|\le\gamma^h C_N/[1+(\gamma^h|x-y|)^N]$ for any $N>1$, if C_N is a suitable constant.

A naive definition for the effective potentials (but, as we will see below, not the most suitable one) could be

$$e^{-V^h(\psi^{\le h})}=\int\prod_{l=h+1}^{0}\{\mathcal{D}\psi^{(l)}e^{-\int dk'\psi^{+(l)}_{k',\sigma}f_h^{-1}G(k')^{-1}\psi^{-(l)}_{k',\sigma}}\}\times e^{-V^0(\psi^{(\le 0)})},\tag{21}$$

where, from now on, we denote by $\psi^{\pm(l)}_{k',\sigma}$ the vector $(\psi^{\pm(l)}_{k'+p_F,1,\sigma},\psi^{\pm(l)}_{k'-p_F,-1,\sigma})$ and

$$G(k')^{-1}=\begin{pmatrix}(-ik_0-\mathbf{k}'^2)-(2\pi\mathbf{k}') & 0\\ 0 & (-ik_0-\mathbf{k}'^2)+(2\pi\mathbf{k}').\end{pmatrix}.\tag{22}$$

One can verify that V^h is given by the sum of terms of the form

$$V^h(\psi^{(\le h)})=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\int\prod_{i=1}^{2m}dk_i' f^h_{n,m}(k_1',\ldots,k_{2m}';z)\times\prod_{i=1}^{m}\psi^{+(\le h)}_{k_i'+\boldsymbol{\omega}_i p_F,\sigma_i}\prod_{i=m+1}^{2m}\psi^{-(\le h)}_{k_i'+\boldsymbol{\omega}_i p_F,\sigma_i}\times\delta\left(\sum_{i=1}^{m}(k_i'+\boldsymbol{\omega}_i p_F)-\sum_{i=m+1}^{2m}(k_i'+\boldsymbol{\omega}_i p_F)+2n\pi\right)\tag{23}$$

and $f^h_{n,m}$, the kernels of the effective potential, are expressed by sum of suitable Feynmann diagrams. A kth order diagram contributing to $f^h_{n,m}$ can be obtained[17] from k graph elements representing the addends in Eq. (18), formed by vertices with emerging oriented half lines with indices $\boldsymbol{\omega}_i$, k_i, h_i symbolizing the fields $\psi^{\pm(h_i)}_{k_i'+\boldsymbol{\omega}_i p_F,\boldsymbol{\omega}_i}$, by pairing the half lines with consistent orientation and same indices $h_i>h$ (contractions) in such a way that the resulting graph turns out to be connected; $2m$ half lines with $h_i\le h$ are left not paired. The not paired lines are called external lines. To each paired line we associate a propagator $g^{h_i}_{\boldsymbol{\omega}_i}(k_i'+\boldsymbol{\omega}_i p_F)$ and to the vertices are associated the kernels in Eq. (18); integrating the product of these factors over all the momenta k_i of the paired lines we obtain the value of the graph contributing to $f^h_{n,m}$, if the expression is multiplied by a suitable sign to take into account the Fermi statistic.

A maximal connected subset of lines with scales $\ge h_v$ is called cluster[17] with scale h_v, and denoted by v. An inclusion relation can be established between the clusters, in such a way that the innermost clusters are the clusters with the higher scale, and so on; see Fig. 1 for an example of graphs with its clusters, pictured as boxes including the paired lines.

Fixing the form of a graph over the sum of all the internal scales gives a contribution to the effective potential. If one excludes the clusters with two or four external lines this sum

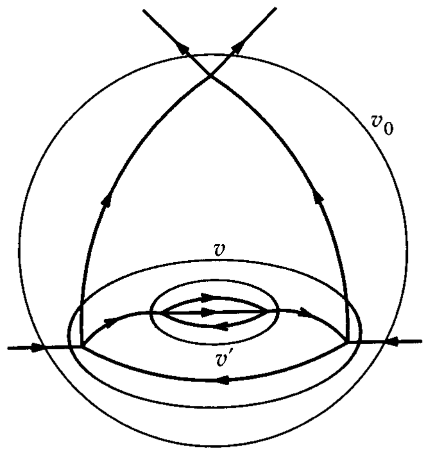

FIG. 1. An example of a graph contributing to $V^{h_{v_0}}$. The closed lines without arrows represent the cluster.

admits a bound uniform in the order of the graph or in L,β; more precisely, the clusters v with two or four external lines which, if present in a graph, make it impossible to obtain a bound uniform in the order of the graph or in L,β are only the ones whose external lines verify the condition[5] $\Sigma_i \varepsilon_i \boldsymbol{\omega}_i p_F + 2n\pi = 0$, where $\boldsymbol{\omega}_i, \varepsilon_i = \pm$ are the indices of the external lines, so that the momenta of the external lines can be simultaneously near the Fermi surface. This is not surprising, since after the sum over the intermediate scales a graph corresponds to a term of the naive unrenormalized perturbation expansion, which by a power counting is a renormalizable theory.

We therefore have to set up a different perturbative expansion for computing $\mathcal{N}$ and this is done, in the renormalization-group framework by introducing a localization operator[17] which is a linear operator acting on the kernels of the effective potential V^h and extracting its relevant part $\mathcal{L}V^h$. The localization operator is defined in the following way:

(1) if $m>4$,

$$\mathcal{L}f^h_{n,m}(k'_1,\ldots,k'_m)=0$$

(2) if $m=4$,

$$\mathcal{L}f^h_{n,4}(k'_1,k'_2,k'_3,k'_4)=\delta_{(\boldsymbol{\omega}_1+\boldsymbol{\omega}_2-\boldsymbol{\omega}_3-\boldsymbol{\omega}_4)p_F+n2\pi,0}f^h_{n,4}(0,0,0,0) \tag{24}$$

(3) if $m=2$,

$$\mathcal{L}f^h_{n,2}(k'_1,k'_2)=\delta_{(\boldsymbol{\omega}_1-\boldsymbol{\omega}_2)p_F+2n\pi,0}[f^h_{n,2}(0,0)+E(\mathbf{k}'+\boldsymbol{\omega}p_F)\partial_{\mathbf{k}'}f^h_{n,2}(0,0)+k^0\partial_{k_0}f^h_{n,2}(0,0)]. \tag{25}$$

The presence of the Kronecker δ's in the right-hand side (rhs) of Eqs. (24) and (25) says that the only relevant processes involve fermions with momenta near the Fermi surface. The relevant part of the effective potential $\mathcal{L}V^h$ depends on the value of p_F and on the value of the spin. In the filled-band case we find

$$\mathcal{L}V^h=\gamma^h n_h F^{\leq h}_\nu+\gamma^h s_h F^{\leq h}_\sigma+a_h F^{\leq h}_\alpha+z_h F^{\leq h}_z+t_h F^{\leq h}_t+i_h F^{\leq h}_\vartheta+\sum_{i=1}^{6}\lambda_{h,i}F^{\leq h}_{\lambda,i}, \tag{26}$$

where $v_h=(n_h,s_h,a_h,z_h,i_h,t_h,\lambda_{h,i})$ (Ref. 27) are called running coupling constants and

$$F^{\leq h}_i=\sum_{\boldsymbol{\omega},\sigma}\int dk'\,\gamma_i\psi^{+(\leq h)}_{k'+\boldsymbol{\omega}p_F,\boldsymbol{\omega},\sigma}\psi^{-(\leq h)}_{k'+\varepsilon_i\boldsymbol{\omega}p_F,\varepsilon_i\boldsymbol{\omega},\sigma}$$

with $\gamma_\nu=1$, $\varepsilon_\nu=1$; $\gamma_\sigma=1$, $\varepsilon_\sigma=-1$; $\gamma_\alpha=E(\mathbf{k}'+\boldsymbol{\omega}p_F)$, $\varepsilon_\alpha=1$; $\gamma_\vartheta=E(\mathbf{k}'+\boldsymbol{\omega}p_F)$, $\varepsilon_\vartheta=-1$; $\gamma_z=-ik_0$, $\varepsilon_z=1$; $\gamma_t=-ik_0$, $\varepsilon_t=-1$; moreover,

$$F^{\leq h}_{\lambda,i}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\boldsymbol{\omega}_3,\boldsymbol{\omega}_4)=\sum_{\sigma,\sigma'}\int\prod_{j=1}^{4}dk'_j\,\psi^{+(\leq h)}_{k'_1+\boldsymbol{\omega}_1p_F,\boldsymbol{\omega}_1,\sigma}\times\psi^{+(\leq h)}_{k'_2+\boldsymbol{\omega}_2p_F,\boldsymbol{\omega}_2,\sigma'}\psi^{-(\leq h)}_{k'_3+\boldsymbol{\omega}_3p_F,\boldsymbol{\omega}_3,\sigma'}\times\psi^{-(\leq h)}_{k'_4+\boldsymbol{\omega}_4p_F,\boldsymbol{\omega}_4,\sigma}\delta\left(\sum_i\varepsilon_i k'_i\right)$$

and, if the fermions are spinning,

$$\begin{aligned}F^{\leq h}_{\lambda,1}&=\sum_{\boldsymbol{\omega}}F^{\leq h}(\boldsymbol{\omega},-\boldsymbol{\omega},\boldsymbol{\omega},-\boldsymbol{\omega}),\\ F^{\leq h}_{\lambda,2}&=\sum_{\boldsymbol{\omega}}F^{\leq 0}(\boldsymbol{\omega},-\boldsymbol{\omega},-\boldsymbol{\omega},\boldsymbol{\omega}),\\ F^{\leq h}_{\lambda,3}&=\sum_{\boldsymbol{\omega}}F^{\leq h}(\boldsymbol{\omega},\boldsymbol{\omega},-\boldsymbol{\omega},-\boldsymbol{\omega}),\\ F^{\leq h}_{\lambda,4}&=\sum_{\boldsymbol{\omega}}F(\boldsymbol{\omega},\boldsymbol{\omega},\boldsymbol{\omega},\boldsymbol{\omega}),\\ F^{\leq h}_{\lambda,5}&=\sum_{\boldsymbol{\omega}}F^{\leq h}(\boldsymbol{\omega},\boldsymbol{\omega},-\boldsymbol{\omega},\boldsymbol{\omega}),\\ F^{\leq h}_{\lambda,6}&=\sum_{\boldsymbol{\omega}}F^{\leq h}(-\boldsymbol{\omega},\boldsymbol{\omega},\boldsymbol{\omega},\boldsymbol{\omega}).\end{aligned} \tag{27}$$

Note the following.

(1) If the fermions are spinless and $p_F\neq\pi/a$ (not filled-band case) $F^{\leq h}_{\lambda,1}\equiv-F^{\leq h}_{\lambda,2}$, $F^{\leq h}_{\lambda,3}=F^{\leq h}_{\lambda,4}=F^{\leq h}_{\lambda,5}=F^{\leq h}_{\lambda,6}=0$, and $g_{1,h}=-g_{2,h}=\lambda_h$, $s_h=t_h=i_h=0$, having also used the Kronecker δ's in Eqs. (24) and (25). Then the relevant part of the effective potential of the model Eq. (1) coincides with the relevant part of the effective potential of the Luttinger model [but in the Luttinger model, by symmetry reason, $n_h=0$ (Ref. 20)].

(2) If the fermions are spinless and $p_F=\pi/a$ (filled-band case) $F^{\leq h}_{\lambda,1}\equiv-F^{\leq h}_{\lambda,2}$, $F^{\leq h}_{\lambda,3}=F^{\leq h}_{\lambda,4}=F^{\leq h}_{\lambda,5}=F^{\leq h}_{\lambda,6}=0$, and $g_{1,h}=-g_{2,h}=\lambda_h$, having used also the Kronecker δ's in Eqs. (24) and (25). Then the relevant part of the effective potential of the model Eq. (1) coincides with the relevant

part of the effective potential of the massive Luttinger model (but in the massive Luttinger model, by symmetry reason, $n_h = t_h = i_h = 0$).

(3) If the fermions are spinning and $p_F \neq \pi/a$, $\pi/2a$, i.e., in the not filled nor half-filled-band case, then by the δ's in Eqs. (24) and (25) $g_{3,h} = g_{5,h} = g_{6,h} = s_h = i_h = t_h = 0$ and the relevant part of the effective potential of the model Eq. (1) coincides with the relevant part of the effective potential of the Luther-Emery model. The terms $F_{\lambda,1}^{\leq h}$ is called in the literature forward scattering and the terms $F_{\lambda,2}^{\leq h}$, $F_{\lambda,4}^{\leq h}$ are called backward scattering.

(4) If the fermions are spinning and $p_F = \pi/2a$, i.e., in the half-filled band case everything is identical to the above case but there is more the Umklapp scattering term $g_{3,h}$. The relevant part of the effective potential of the model Eq. (1) coincides in this case with the relevant part of the effective potential of the Umklapp model.

(5) Finally, if the fermions are spinning and $p_F = \pi/a$, i.e., in the filled-band spinning case all the terms in Eq. (26) are present. The model corresponding to this case has no counterpart in the g-ology literature.

Note then, that our renormalization-group treatment shows in a very clear way the relation between the model Eq. (1) and the g-ology models usually studied in the literature.

By means of the localization operator we define a different perturbative expansion for computing $\mathcal{N}$. If $\mathcal{R} = 1 - \mathcal{L}$ and $C_h^{-1} \equiv C_h^{-1}(k') = \Sigma_{k=-\infty}^{h} f_k(k')$ we write

$$\int \{\mathcal{D}\psi^{(\leq 0)} e^{-\int dk' \psi^{+(\leq 0)}_{k',\sigma} C_0 G^{-1} \psi^{-(\leq 0)}_{k',\sigma}}\} e^{-\mathcal{L}V^0(\psi^{(\leq 0)}) - \mathcal{R}V^0(\psi^{(\leq 0)})}$$
$$= \int \{\mathcal{D}\psi^{(\leq -1)} e^{-\int dk' \psi^{+(\leq -1)}_{k',\sigma} C_{-1} G^{-1} \psi^{-(\leq -1)}_{k',\sigma}}\} e^{-V^{-1}(\psi^{(\leq -1)})}$$
$$= \int \{\mathcal{D}\psi^{(\leq -1)} e^{-\int dk \psi^{+(\leq -1)}_{k',\sigma} C_{-1} G^{-1} \psi^{-(\leq -1)}_{k',\sigma}}\} e^{-\mathcal{L}V^{-1}(\psi^{(\leq -1)}) - \mathcal{R}V^{-1}(\psi^{(\leq -1)})} = \cdots, \tag{28}$$

i.e., we organize each integration by writing the effective potential V^h as $\mathcal{L}V^h + \mathcal{R}V^h$. In this way V^h is a series (renormalized expansion) in the running-coupling constants $v_k = (n_h, s_h, a_h, z_h, i_h, t_h, \lambda_{i,h})$, $k > h$ and v_h obeys to a recursive relation, called beta function, $v_h = \beta_h(v_{h+1}, \ldots, v_0)$.[17,19]

The effective potential is still given by a sum of Feynmann graphs which differ with respect to others previously introduced because they are obtained contracting the half lines of vertices not only coming from the addends in Eq. (18) but also from the addends in $\mathcal{L}V^k$, with $k \geq h$; moreover on each cluster v with two or four external lines the $\mathcal{R}$ operation acts so that its value $f_2^{h_v}$ or $f_4^{h_v}$ in the graph is replaced by $\mathcal{R}f_2^{h_v}$ and $\mathcal{R}f_4^{h_v}$; it is easy to check[6] that the sum over the intermediate scales of these graphs is bounded uniformly in the order of the graph as well as in L, β. Even more, estimating the fermionic expectations in the renormalized expansion for the effective potential by the Grahm-Hadamard inequality (which exploits the anticommutativity of fermions), one can prove the analyticity of the effective potential as a function of the running-coupling constants in a small domain. This follows from Ref. 28 in which a renormalized expansion for the effective potential essentially identical to our one is discussed for the Gross-Neveu model. Of course the above procedure is a resummation of the previously, apparently divergent series. Therefore, the difficulties are now hidden in the running-coupling constants. To be useful, the new expansion for the effective potential requires knowledge of the h dependence of the running-coupling constants, and, in particular, that $\max_{k>h}|v_k|$ is so small that the series converge. The h dependence of the running-coupling constants can be derived by the study of the beta function. However, we do not expect that $\max_{k>h}|v_k|$ is small as is noted below.

(1) We have assumed that $\gamma^h n_h$, $\gamma^h s_h$ are vanishing as $h \to -\infty$. One can expect to obtain this for the running-coupling constants corresponding to $F_\nu^{\leq h}$, by choosing in a suitable way the counterterm ν, but there are no free parameters in the Hamiltonian for the running-coupling constants corresponding to $F_\sigma^{\leq h}$.

(2) Also, the marginal couplings give problems. In fact by a second-order computation we get
$$a_{h-1} = a_h + \beta_1 \lambda_h^2, \quad z_{h-1} = z_h + \beta_1 \lambda_h^2, \qquad \lambda_{h-1} = \lambda_h, \tag{29}$$
with $\beta_1 > 0$, so that the vanishing of the second-order β function for λ_h has the effect that λ_h is constant in this approximation and therefore a_h, z_h grows more and more as $h \to -\infty$. One could hope that the third-order β function is not vanishing and negative, but indeed an explicit computation shows that it has the effect of increasing λ_h.[29]

IV. ANOMALOUS SCALING

The fact that, at least by a second-order computation, the running-coupling constants seem to be unbounded (or at least they grow so large to go beyond the possibility of our nonperturbative approach) is an indication that the Schwinger function decay for large distances is different in the free or interacting theory. It is possible to extend the methods followed so far to a more general approach which can take into account a possible modification of the Schwinger function behavior with respect to the free case. Such behavior is called ‘‘anomalous scaling.’’[30,31,19–21,5,6,17]

The idea is to integrate over the fields on each scale by inserting the terms of the relevant part of the effective potential, that in the previous approach causes the uncontrollable growth of the running-coupling constants, in the integration, so taking them into account by a change in the propagator. For simplicity of notations we discuss the spinless case, but everything in this section holds for any value of σ.

The integration can be defined recursively in the following way, setting $Z_0=1$: once the fields $\psi^0,\ldots,\psi^{h+1}$ have been integrated we have to evaluate

$$\mathcal{N}=\int \{\mathcal{D}\psi^{(\leq h)}e^{-\int dk' C_h Z_h \psi^{+(\leq h)}_{k',\sigma}\mathcal{G}^{(h)}(k')^{-1}\psi^{-(\leq h)}_{k',\sigma}}\} \times e^{-V^h(\sqrt{Z_h}\psi^{(\leq h)})}, \tag{30}$$

where

$$\mathcal{L}V^h=n_hF_\nu^{\leq h}+s_hF_\sigma^{\leq h}+a_hF_\alpha^{\leq h}+z_hF_z^{\leq h}+i_hF_t^{\leq h}+t_hF_\vartheta^{\leq h}+\lambda_hF_\lambda^{\leq h},$$

and

$$\mathcal{G}^{(h)}(k')^{-1}=\begin{pmatrix}(-ik_0-\mathbf{k}'^2-2\pi\mathbf{k}') & \sigma_h(k')\\ \sigma_h(k') & (-ik_0-\mathbf{k}'^2+2\pi\mathbf{k}')\end{pmatrix}. \tag{31}$$

We write Eq. (30) as

$$\int \{\mathcal{D}\psi^{(\leq h)}e^{-\int dk' \psi^{+(\leq h)}_{k',\sigma}C_h(k')Z_{h-1}(k')\mathcal{G}^{(h-1)}(k')^{-1}\psi^{-(\leq h)}_{k',\sigma}}\} \times e^{-\overline{V^h}(\sqrt{Z_h}\psi(\leq h)}, \tag{32}$$

where $\widetilde{V}^h=\mathcal{L}\overline{V^h}+(1-\mathcal{L})V^h$,

$$\mathcal{L}\widetilde{V}^h=n_hF_\nu^{\leq h}+(a_h-z_h)F_\alpha^{\leq 0}+t_hF_t^{\leq 0}+i_hF_\vartheta^{\leq h}+\lambda_hF_\lambda^{\leq h},$$

and

$$Z_{h-1}(k')=Z_h+C_h^{-1}(k')Z_hz_h,\qquad Z_{h-1}(k')\sigma_{h-1}(k')=Z_h\sigma_h(k')+Z_hC_h^{-1}(k')s_h. \tag{33}$$

Now one can perform the integration respect to $\psi^{(h)}$ writing Eq. (32) as

$$\int \{\mathcal{D}\psi^{(\leq h-1)}e^{-\int dk' \psi^{+(\leq h-1)}_{k',\sigma}C_{h-1}Z_{h-1}\mathcal{G}^{(h-1)}(k')^{-1}\psi^{-(\leq h-1)}_{k',\sigma}}\} \times \int \{\mathcal{D}\psi^{(h)}e^{-\int dk' \psi^{+(h)}_{k',\sigma}\widetilde{f}_hZ_{h-1}\mathcal{G}^{(h-1)}(k')^{-1}\psi^{-(h)}_{k',\sigma}}\} \times e^{-\mathcal{L}\overline{V^h}(\sqrt{Z_{h-1}}\psi^{(\leq h)})+\mathcal{R}\overline{V^h}(\sqrt{Z_{h-1}}\psi^{(\leq h)})}, \tag{34}$$

where

$$\widetilde{f}_h\equiv\widetilde{f}_h(k')=Z_{h-1}\left[\frac{C_h^{-1}(k')}{Z_{h-1}(k')}-\frac{C_{h-1}^{-1}(k')}{Z_{h-1}}\right],$$

$$Z_{h-1}\equiv Z_{h-1}(0),\quad \mathcal{R}\hat{V}^h(\psi)=\mathcal{R}V^h\left[\left(\frac{Z_h}{Z_{h-1}}\right)^{1/2}\psi\right],$$

and

$$\mathcal{L}\hat{V}^h=\gamma^h\nu_hF_\nu^{\leq h}+\delta_hF_\alpha^{\leq h}+\tau_hF_t^{\leq h}+\vartheta_hF_\vartheta^{\leq h}+g_hF_\lambda^{\leq h}, \tag{35}$$

with

$$\gamma^h\nu_h=\frac{Z_h}{Z_{h-1}}n_h,\quad \delta_h=\frac{Z_h}{Z_{h-1}}(a_h-z_h),\quad \tau_h=\frac{Z_h}{Z_{h-1}}t_h,\quad \vartheta_h=\frac{Z_h}{Z_{h-1}}i_h,\quad g_h=\left(\frac{Z_h}{Z_{h-1}}\right)^2\lambda_h \tag{36}$$

and, if

$$\mathcal{G}^{h-1}(k')=\frac{1}{A_{h-1}(k')}\begin{pmatrix}(-ik_0-\mathbf{k}'^2+2\pi\mathbf{k}') & -\sigma_{h-1}(k')\\ -\sigma_{h-1}(k') & (-ik_0-\mathbf{k}'^2-2\pi\mathbf{k}')\end{pmatrix},$$

$$A_{h-1}(k')=[(-ik_0+\mathbf{k}'^2)^2-(2\pi\mathbf{k}')^2-\sigma_{h-1}(k')^2],$$

the anomalous propagator corresponding to the second integration in Eq. (34) is

$$g^h(x-y)=\frac{1}{Z_{h-1}}\int dke^{ik(x-y)}\widetilde{f}_h(k)\mathcal{G}^{(h-1)}(k). \tag{37}$$

We perform the integration with respect to the field $\psi^{(h)}$ obtaining

$$\int \{\mathcal{D}\psi^{(\leq h-1)}e^{-\int dk' \psi^{+(\leq h-1)}_{k',\sigma}C_{h-1}(k')Z_{h-1}\mathcal{G}^{(h-1)}(k)^{-1}\psi^{+(\leq h-1)}_{k',\sigma}}\} \times e^{-V^h(\sqrt{Z_{h-1}}\psi^{(\leq h-1)})} \tag{38}$$

and the procedure can be iterated. Note that $\sigma_h(k')$ is ‘‘weakly’’ dependent on k'. The effective potential $V^h(\sqrt{Z_{h-1}}\psi^{(\leq h-1)})$ is given by a sum of Feynmann graphs similar to the one introduced in the preceding section for the renormalized expansion of the effective potential, with the difference that to the paired lines with scale k joining two vertices with scales k_1 and k_2, the factor $\sqrt{Z_{k_1}Z_{k_2}}g^{(k)}$, if $g^{(k)}$ is the anomalous propagator Eq. (37), is associated; moreover in $\mathcal{L}V^k$ there are no vertices corresponding to the running-coupling constants which in the previous approach grew, so affecting the convergence of the series.

If $\sigma_h\equiv\sigma_h(0)$, one can check[6] that the anomalous propagator Eq. (37) is bounded by, for any $N>1$

$$|g^{(h)}(x-y)|\leqslant A \max_{i=0,1,2}\left|\frac{\sigma_h}{\gamma^h}\right|^i \frac{\gamma^h}{Z_h}\frac{C_N}{1+(\gamma^h|x-y|)^N}, \quad (39)$$

where A is a suitable constant matrix and C_N is a suitable constant. In the not filled-band case[5] one introduces an anomalous-scaling integration identical to the one discussed so far with the only difference that $\vartheta_h=\tau_h=\sigma_h\equiv 0$, see the considerations after Eq. (26). On the other hand, the anomalous propagator obeys the same bound Eq. (39), with $\max_{i=0,1,2}|\sigma_h/\gamma^h|^i$ replaced by a constant not depending on h. In the not filled-band case it was proved[21,5] that, if

$$\max_{k\geqslant h}|\nu_k||\overline{v_k}|=\overline{\varepsilon_h}, \quad \max_{k\geqslant h}\left|\frac{Z_k}{Z_{k-1}}\right|\leqslant e^{C\overline{\varepsilon}_h^2}, \quad (40)$$

with $\overline{v_k}=(\delta_h,\tau_h,\vartheta_h,g_h)$, the k-order contributions to the kernels of the effective potential V^{h-1} and of the β function β^{h-1} is bounded by $C^k\overline{\varepsilon}_h^k$; the same proof holds in this case (up to ''trivial'' changes) and it allows us to conclude that the k-order contributions to the kernels of the effective potential or to the β function is bounded by $C^k\tilde{C}_h^k\varepsilon_h^k$, if

$$\tilde{C}_h=\max_{\substack{i=0,1,2\\ k\geqslant h}}\left|\frac{\sigma_k}{\gamma^k}\right|^i.$$

This bound does not depend on L and we can remove the infrared cutoff, i.e., we can take the limit $L\to-\infty$ on the effective potential.

V. THE FLOW OF THE RENORMALIZATION GROUP

Contrary to the preceding sections, in the following, the assumption of spinless fermions is crucial. We have seen at the end of Sec. IV that, given a constant $\tilde{C}$, if $\tilde{C}_h\leqslant\tilde{C}$ the kernels of the effective potential V^{h-1} and the beta function β^{h-1} are analytic as functions of their arguments, if $\overline{\varepsilon_h}<\varepsilon<1/\tilde{C}C$ and $\overline{\varepsilon_h}$ is defined in Eq. (40).[32] Of course, there is no reason for which $\tilde{C}_h\leqslant\tilde{C}$ for any h so we call $h^*\equiv\tilde{h}(\tilde{C})$ the scale such that $\tilde{C}_{h^*}\leqslant\tilde{C}$ and $\tilde{C}_{h^*-1}>\tilde{C}$; the scale h^* can be computed once we know the h dependence of σ_h. Then, if for $h\geqslant h^*$ it holds that $\overline{\varepsilon_h}<\varepsilon$ we have that the kernels of the effective potential V^{h-1} and the beta function β^{h-1} are well defined for $h\geqslant h^*$; to prove this, and compute h^* we need information on the h dependence of the running-coupling constants, what is provided by the study of the beta function.

It is possible to choose the counterterm ν so that $|\nu_h|<\varepsilon$ for any $0\geqslant h\geqslant h^*$; in fact, given any sequence of running-coupling constants verifying $\max_{i,k\geqslant h}|\overline{v_k}|\leqslant\varepsilon$, $\max_{k\geqslant h}|Z_k/Z_{k-1}|\leqslant e^{\beta_1\varepsilon}$, ν_h obeys to the equation $\nu_{h-1}=\gamma\nu_h+\beta_\nu^h$, with $|\beta_\nu^h|<K\varepsilon^2$, if K is a constant. One obtains that $\nu_h=\gamma^{-h}(\nu_0+\sum_{j=h+1}^0\gamma^{j-1}\beta_\nu^j)$ so that, choosing ν such that ν_0 satisfies $\nu_0+\sum_{j=h^*}^0\gamma^{j-1}\beta_\nu^j=0$ of course $\nu_{h^*}=0$ and $|\nu_h|\leqslant\varepsilon$ for any $h\geqslant h^*$. The value of ν_0 so that $|\nu_h|\leqslant\varepsilon$ for any $h\geqslant h^*$ is not unique; from the value of h^* computed in Eq. (44) it is clear that, if $\overline{\nu_0}$ is the value such that $\nu_{h^*}=0$, any $\nu_0=\overline{\nu_0}+O(u^2)$ has the effect that $|\nu_h|\leqslant\varepsilon$ for any $h\geqslant h^*$.

It is convenient to write the anomalous propagator Eq. (37) as[6]

$$g^h_{\boldsymbol{\omega},\boldsymbol{\omega}}(x-y)=g^h_{\boldsymbol{\omega},L}(x-y)+C^h_{1,\boldsymbol{\omega}}(x-y)+C^h_{2,\boldsymbol{\omega}}(x-y), \quad (41)$$

$$g^h_{\boldsymbol{\omega},L}(x-y)=\int\frac{dk'}{(2\pi)^2}\frac{e^{ik'x}}{Z_h}\frac{f[\gamma^{-2h}(k_0^2+\mathbf{k}'^2)]}{-ik_0-2\pi\boldsymbol{\omega}\mathbf{k}'},$$

where $g^{(h)}_{\boldsymbol{\omega},L}(x-y)$ is just the propagator ''at scale h'' of the Luttinger model[20] and

$$|g^{(h)}_{\boldsymbol{\omega},L}(x-y)|\leqslant B\frac{\gamma^h}{Z_h}\frac{C_N}{1+(\gamma^h|x-y|)^N},$$

for a suitable constant B. Moreover,

$$|C_1(x-y)|\leqslant B\frac{\gamma^{2h}}{Z_h}\frac{C_N}{1+(\gamma^h|x-y|)^N},$$

$$|C_2(x-y)|\leqslant B\frac{\gamma^h}{Z_h}\left(\frac{\sigma_h}{\gamma^h}\right)^2\frac{C_N}{1+(\gamma^h|x-y|)^N},$$

and

$$|g^{(h)}_{\boldsymbol{\omega},-\boldsymbol{\omega}}(x-y)|\leqslant B\frac{\gamma^h}{Z_h}\frac{\sigma_h}{\gamma^h}\frac{C_N}{1+(\gamma^h|x-y|)^N}.$$

This decomposition of the propagator will allow us to extract in the β function a part coinciding with the Luttinger model β function. In fact we can write

$$\begin{aligned}
g_{h-1}&=g_h+G_\lambda^{1,h}+G_\lambda^{2,h}+\gamma^hR_\lambda^h,\\
\sigma_{h-1}&=\sigma_h+G_\sigma^{1,h}+\gamma^hR_\sigma^h,\\
\delta_{h-1}&=\delta_h+G_\delta^{1,h}+G_\delta^{2,h}+\gamma^hR_\delta^h,\\
\tau_{h-1}&=\tau_h+G_\tau^{2,h}+\gamma^hR_\tau^h,\\
\vartheta_{h-1}&=\vartheta_h+G_\vartheta^{2,h}+\gamma^hR_\theta^h,\\
\frac{Z_{h-1}}{Z_h}&=1+G_z^{1,h}+G_z^{2,h}+\gamma^hR_z^h,
\end{aligned} \quad (42)$$

where (1) $G_\lambda^{1,h}\equiv G_\lambda^{1,h}(g_h,\delta_h;\ldots;g_0,\delta_0)$, $G_\delta^{1,h}\equiv G_\delta^{1,h}(g_h,\delta_h;\ldots;g_0,\delta_0)$ and $G_z^{1,h}\equiv G_z^{1,h}(g_h,\delta_h;\ldots;g_0,\delta_0)$ are given by series of terms involving only the Luttinger model part of the propagator $g^k_{\boldsymbol{\omega},L}(x-y)$, $k\geqslant h$, see Eq. (41); (2) $G_\sigma^{1,h}$, $G_\lambda^{2,h}$, $G_\delta^{2,h}$, $G_z^{2,h}$, $G_\tau^{2,h}$, $G_\vartheta^{2,h}$ depend on all the running-coupling constants and are given by a series of terms involving at least a propagator $C^k_{2,\boldsymbol{\omega}}(x-y)$ or $g^{(k)}_{\boldsymbol{\omega},-\boldsymbol{\omega}}(x-y)$, $k\geqslant h$; (3) R_i^h, $i=\lambda$, z, σ, δ, τ, ϑ depend on all the running-coupling constants and are given by a series of terms involving at least a propagator $C^k_{1,\boldsymbol{\omega}}(x-y)$, $k\geqslant h$. We have not written the β function for ν_h, as we know that with the right choice of ν we have that $|\nu_h|\leqslant\varepsilon$ for any $h\geqslant h^*$.

The β function generates a recursion that is a short memory dynamical system in the sense that is a set of equations of the form $v_{h-1}=\beta_h(v_h,v_{h+1},\ldots,v_0)$ which behaves ''essentially'' as a system without memory $v_{h-1}=\beta_h(v_h,v_h,\ldots,v_h)$ (this is a consequence of the convergence of the β function as function of its arguments[21]). Even more, the convergence of Eq. (42) allows us to say that the lowest nonzero terms determinate the evolution of the running-coupling constants.

Let us call $\{G^h\}_2$ the second-order contribution to the β function with equal arguments, $G^h \equiv G^h(v_h,\ldots,v_h)$. Then by an explicit computation

$$\{G_\lambda^{1,h}\}_2=0 \quad \{G_\delta^{1,h}\}_2=0,$$

$$\{G_z^{1,h}\}_2=\beta_3 g_h^2, \quad \{G_\sigma^{1,h}\}_2=-\beta_1 g_h \sigma_h,$$

$$\{G_\tau^{2,h}\}_2=\beta_4 g_h \frac{\sigma_h}{\gamma^h}, \quad \{G_\tau^{2,h}\}_2=\beta_5 g_h \frac{\sigma_h}{\gamma^h},$$

with $\beta_1,\ \beta_2,\ \beta_3,\ \beta_4>0$.

The fact that $\{G_\lambda^{1,h}\}_2=0$, $\{G_\delta^{1,h}\}_2=0$ could generate a problem, as one cannot exclude *a priori*, that there is some nonvanishing term at some large order, and the evolution of the running-coupling constants will depend critically on this unknown term. Fortunately, it is possible to prove that

$$\lim_{h\to-\infty} G_\lambda^{1,h}(g,\delta;\ldots;g,\delta)=0,$$

$$\lim_{h\to-\infty} G_\delta^{1,h}(g,\delta;\ldots;g,\delta)=0,$$

i.e., it is vanishing at all orders. In fact $G_\lambda^{1,h}$, $G_\delta^{1,h}$ coincide, up to terms vanishing as $h\to-\infty$ as $O(\gamma^h)$, with the correspondent quantities of the Luttinger model, and, by using the properties of the exact solution[10,12] one can prove that it is vanishing.[17,20,21,5,33]

Finally, as $|R_i^h|\leq K\varepsilon_h^2$, $i=\lambda,\delta,\sigma,z$, and $|G_\lambda^{2,h}|$, $|G_\delta^{2,h}|$, $|G_z^{2,h}|$, $|G_\tau^{2,h}|$, $|G_\vartheta^{2,h}|\leq K\varepsilon_h \tilde{C}_h$ one finds[6]

$$\begin{aligned}
-c_1\lambda^2<|g_{h-1}-g_0|<c_2\lambda^2,\\
\lambda\beta_1 c_4 h\leq \ln\left(\frac{\sigma_{h-1}}{\sigma_0}\right)\leq \lambda\beta_1 c_3 h,\\
-\beta_3 c_1\lambda^2 h\leq \ln(|Z_{h-1}|)\leq -\beta_3 c_2 \lambda^2 h,\\
-c_1|\lambda|<|\tau_{h-1}-\tau_0|<c_2|\lambda|,\\
-c_1|\lambda|<|\vartheta_{h-1}-\vartheta_0|<c_2|\lambda|,\\
-c_1|\lambda|<|\delta_{h-1}-\delta_0|<c_2|\lambda|
\end{aligned} \tag{43}$$

for suitable constants $c_1,c_2>0$ and $c_4,c_3>0$, i.e., the flow is essentially described by the second-order truncation of the β function. We use that $\Sigma_{h=h^*}^0 \tilde{C}_h\leq K$, if K is a constant.

From Eq. (43) it follows that it is possible to choose λ small enough so that the series for V^{h-1} and β^{h-1}, if $h\geq h^*$, are convergent; moreover from Eq. (43) it is possible to obtain an upper and lower bound for h^*

$$\frac{\ln_\gamma(\tilde{C}^{-1}\sigma_0)}{1-\lambda\beta_1 c_4}\leq h^* \leq \frac{\ln_\gamma(\tilde{C}^{-1}\sigma_0)+1}{1-\lambda\beta_1 c_3}. \tag{44}$$

The above analysis is performed by approximating the anomalous propagator Eq. (37) with the Luttinger model propagator by Eq. (41). This approximation is, of course, not reasonable for large $|h|$ corresponding to momenta negligible with respect to the σ_h term due to the periodic potential. But for the scales $h<h^*$ the bound[6]

$$|g^{(<h^*)}(x-y)|\leq A\,\frac{\gamma^{h^*}}{Z_{h^*}}\left(\frac{\gamma^{h^*}}{\sigma_{h^*}}\right)\frac{C_N}{1+\sigma_{h^*}^N|x-y|^N} \tag{45}$$

holds and, from Eqs. (43) and (44), $(\gamma^{h^*}/\sigma_{h^*})\leq K/C$, if K is a constant. In other words, the propagator $g_{\boldsymbol{\omega},\boldsymbol{\omega}'}^{(\leq h^*)}(x-y)$ obeys to the same bound of $g_{\boldsymbol{\omega},\boldsymbol{\omega}'}^{(h)}(x-y)$ for $h\geq h^*$; our choice of h^* is made just to obtain this. The integration of the scale between $-\infty$ and h^* is equivalent to the integration of a single scale in a not filled-band theory, so that for the same considerations at the end of Sec. IV also the series expressing

$$\int \{\mathcal{D}\psi^{(<h^*)} e^{-\int dk\,\psi_{k,\sigma}^{+(<h^*)} C_{h-1}(k) Z_{h-1} \mathcal{G}^{(h^*)}(k)^{-1}\psi_{k,\sigma}^{+(<h^*)}}\} \times e^{-V^{h^*}(\sqrt{Z_{h^*}})\psi^{(\leq h^*)})} \tag{46}$$

is convergent. The nth order of the series in the running-coupling constants for the effective potential for $h>h^*$ is bounded by $(\overline{\varepsilon_{h^*}})^n \tilde{C}^n C_1^n$ while the nth order term of the series Eq. (46) is bounded by $(\overline{\varepsilon_{h^*-1}})^n (C_2/C)^n$ so that there is a nonambiguous way to fix $\tilde{C}$ so that $\overline{\varepsilon_{h^*}}$ is the largest possible.

The Schwinger function Eq. (2) admits a perturbative expansion similar to the one of the partition function, whose convergence follows from the partition function expansion convergence;[21] it holds that

$$S(x,y)=S^{u.v.}(x,y)+\sum_{h=h^*}^{0}\sum_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2} e^{i\pi(\boldsymbol{\omega}_1\mathbf{x}-\boldsymbol{\omega}_2\mathbf{y})} \times[g_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}^{(h)}(x-y)+\overline{S}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}^{(h)}(x,y)], \tag{47}$$

where we call $g^{(\leq h^*)}(x-y)$ simply $g^{(h^*)}(x-y)$ and the first addend is bounded by $C_N/(1+|x-y|^N)$; $\overline{S}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}^{(h)}(x,y)$ is given by a sum of Feynmann graphs similar to the ones contributing to the effective potential, see Sec. III, with the difference that (1) they have two external lines, to which the propagators $g_{\boldsymbol{\omega}_1,\boldsymbol{\omega}'}^{(h_1)}(x-x')$ or $g_{\boldsymbol{\omega}'',\boldsymbol{\omega}_2}^{(h_2)}(y-y')$ are associated, with $h_1,h_2\geq h$ and if x',y' are, respectively, the coordinates of the vertex the external lines are entering in or coming out; (2) h is the smallest scale of the propagators contributing to $\overline{S}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}^{(h)}(x-y)$ (and not, as for the graphs for the effective potential, the scale of the external lines); (3) no $\mathcal{R}$ acts on clusters containing the external lines.

Note that σ_{h^*}, Z_{h^*} depend on $\tilde{C}$, but it is easy to check that they can be written as $\sigma_{h^*}=\overline{\sigma_{h^*}}(1+\lambda f_1)$, $Z_{h^*}=\overline{Z_{h^*}}(1+\lambda f_2)$, with $|f_1|$, $|f_2|$ bounded by some constant and $\overline{\sigma_{h^*}}$, $\overline{Z_{h^*}}$ independent on $\tilde{C}$. Let us define

$$\eta_1=-\frac{\ln(\overline{Z_{h^*}})}{\ln u}, \quad 1+\eta_2=\frac{\ln(\overline{\sigma_{h^*}})}{\ln u} \tag{48}$$

which are, respectively, $O(\lambda^2)$ and $O(\lambda)$, with $\mathrm{sign}(\eta_2)=\mathrm{sign}(\lambda)$.

One can check[21] that $|\overline{S^{(h)}_{\boldsymbol{\omega},\boldsymbol{\omega}}}(x,y)|\leqslant A\tilde{\varepsilon}(\gamma^h/Z_h)C_N/[1+(\gamma^h|x-y|)^N]$ and

$$|\overline{S^{(h)}_{\boldsymbol{\omega},-\boldsymbol{\omega}}}(x-y)|\leqslant A\tilde{\varepsilon}(\sigma_h/Z_h)C_N/1+(\gamma^h|x-y|)^N,$$

if $\tilde{\varepsilon}=\max(u,u^{1+\eta_1},|\lambda|)$; the extra factor σ_h/γ^h in the second bound follows from the fact that in the graph contributing to $\overline{S^{(h)}_{\boldsymbol{\omega},-\boldsymbol{\omega}}}(x,y)$ there is at least a nondiagonal propagator, which obeys, see the lines after Eq. (41), to a bound similar to the diagonal propagator but with a factor more $\sigma_k/\gamma^k\leqslant\sigma_h/\gamma^h$, $k\geqslant h$.

We now prove the statements of Sec. II. The bounds Eqs. (9) and (10) can be obtained by Eq. (47), as for $\gamma^{-h_x-1}<|x-y|\leqslant\gamma^{-h_x}$, $h_x>h^*$, if $|g^{(h)}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}(x-y)+\overline{S^{(h)}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}}(x,y)|\leqslant\overline{g^{(h)}}(x-y)$:

$$\sum_{h=0}^{h^*}\overline{g^{(h)}}(x-y)\leqslant A_1\left[\sum_{h=h^*}^{h_x-1}\frac{\gamma^h}{Z_h}+\sum_{h=h_x}^{0}\frac{\gamma^h}{Z_h}\frac{C_N}{\gamma^{Nh}|x-y|^N}\right]\leqslant A_2\gamma^{h_x(1-\eta_3)} \tag{49}$$

while for $|x-y|\geqslant\sigma_{h^*}^{-1}$:

$$\sum_{h=0}^{h^*}\overline{g^{(h)}}(x-y)\leqslant\frac{C_N}{|x-y|^N}\sum_{h=h^*+1}^{1}\frac{\gamma^{-(N-1)h}}{Z_h}\leqslant A_2\frac{\gamma^{h^*}}{Z_{h^*}}\frac{C_N}{(\gamma^{h^*}|x-y|)^N}. \tag{50}$$

The occupation-number discontinuity can be obtained noting that, from the theory of the Bloch waves, $\phi(p_F^+,\mathbf{x})=\cos(p_F\mathbf{x})+O(u)$ and $\phi(p_F^-,\mathbf{x})=i\sin(p_F\mathbf{x})+O(u)$ so that the occupation-number discontinuity Eq. (5) can be written as

$$Z^{-1}=\frac{1}{L}\int d\mathbf{x}\,d\mathbf{y}\{\cos[p_F(\mathbf{x}+\mathbf{y})]+r(u)\}S^{L,\beta}(x,y)$$

with $r(u)=O(u)$. By inserting in the above expression Eq. (47) we obtain several terms that we can bound in the following way, if K_i are positive constants:

$$\frac{r(u)}{L}\int d\mathbf{x}\,d\mathbf{y}\sum_{h=h^*}^{0}\sum_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}|g^{(h)}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}(x,y)+\overline{S^{(h)}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}}(x,y)|\leqslant K_1u\sum_{h=h^*}^{0}\frac{1}{Z_h}\leqslant K_2u\ln u\leqslant K_3u^{\eta_1}, \tag{51}$$

$$\frac{1}{L}\int d\mathbf{x}\,d\mathbf{y}\cos[p_F(\mathbf{x}+\mathbf{y})]\sum_{h=h^*}^{0}\sum_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}|\overline{S^{(h)}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}}(x,y)|\leqslant K_4\sum_{h=h^*}^{0}\frac{\sigma_h}{Z_h\gamma^h}\leqslant\overline{\varepsilon}K_5u^{\eta_1} \tag{52}$$

in which we take into account that in the sum only the terms $\boldsymbol{\omega}_1=-\boldsymbol{\omega}_2$ survive;

$$\frac{1}{L}\int d\mathbf{x}\,d\mathbf{y}\cos[p_F(\mathbf{x}+\mathbf{y})]g^{(h)}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}(x-y)=\int dk_0\sum_{h=h^*}^{0}\tilde{f}_h(0,k_0)\frac{1}{Z_h}[\mathcal{G}_h(0,k_0)^{-1}]_{\boldsymbol{\omega},-\boldsymbol{\omega}} \tag{53}$$

verifies the bound Eq. (12); finally

$$\frac{1}{L}\int d\mathbf{x}\,d\mathbf{y}\{\cos[p_F(\mathbf{x}+\mathbf{y})]+r(u)\}S^{u.v.}(x,y)\leqslant K_7u. \tag{54}$$

Let us prove the decomposition Eq. (14). By diagonalizing the quadratic form $\hat{g}^{(h)}(x,y)=\sum_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}e^{i\pi(\boldsymbol{\omega}_1\mathbf{x}-\boldsymbol{\omega}_2\mathbf{y})}g^{(h)}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}(x-y)$ it is possible to see that

$$\hat{g}^{(h)}(x,y)=\int dk'e^{ik'(x-y)}\frac{\tilde{f}_h(k')}{Z_h}\left[\frac{F_{xy}(k',\sigma_h)}{A+B}+\frac{F_{xy}(-k',-\sigma_h)}{A-B}\right],$$

where

$$F_{xy}(k',\sigma_h)=\hat{\phi}(k',\mathbf{x},\sigma_h)\hat{\phi}(k',-\mathbf{y},\sigma_h),$$

$$\hat{\phi}(k',\mathbf{x},\sigma_h)=\frac{1}{\sqrt{2B}}\left[\sqrt{B-C}e^{ip_F\mathbf{x}}-\frac{e^{-ip_F\mathbf{x}}}{\sqrt{B-C}}\right],$$

and $A=-ik_0+\mathbf{k}'^2$, $B=\sqrt{(C^2+\sigma_h^2)}$ and $C=2\pi\mathbf{k}'$. We can rewrite the above integral in terms of the k variable. Recall that, if $\boldsymbol{\omega}=\text{sign}(\mathbf{k})$ and $f_h(k')\neq0$, then $k=\boldsymbol{\omega}p_F+\mathbf{k}'$. Hence

$$\hat{g}^h(x-y)=\sum_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}e^{i\pi(\boldsymbol{\omega}_1\mathbf{x}-\boldsymbol{\omega}_2\mathbf{y})}g^h_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}(x-y)=\int dk\frac{\tilde{f}_h(k)}{Z_h(k)}\frac{\hat{\phi}(\mathbf{k},\mathbf{x},\sigma_h)\hat{\phi}(\mathbf{k},-\mathbf{y},\sigma_h)e^{ik_0(x_0-y_0)}}{-ik_0-[\hat{\varepsilon}(k,\sigma_h)-\pi^2]}, \tag{55}$$

where

$$\hat{\varepsilon}(k,\sigma_h)=(|\mathbf{k}|-\pi)^2+2\pi\,\text{sign}(|\mathbf{k}|-\pi)\times\sqrt{(|\mathbf{k}|-\pi)^2+\sigma_h^2}+\pi^2 \tag{56}$$

and

$$\hat{\phi}(\mathbf{k},\mathbf{x},\sigma_h)=e^{i\mathbf{k}\mathbf{x}}u(\mathbf{k},\mathbf{x},\sigma_h), \tag{57}$$

$$u(\mathbf{k},\mathbf{x},\sigma_h)=e^{-i\,\text{sign}(\mathbf{k})\pi\mathbf{x}}\times\left[\cos(\pi\mathbf{x})\sqrt{1+\frac{\text{sign}(|\mathbf{k}|-\pi)\sigma_h}{\sqrt{(|\mathbf{k}|-\pi)^2+\sigma_h^2}}}+i\,\text{sign}(\mathbf{k})\sin(\pi\mathbf{x})\sqrt{1-\frac{\text{sign}(|\mathbf{k}|-\pi)\sigma_h}{\sqrt{(|\mathbf{k}|-\pi)^2+\sigma_h^2}}}\right]. \tag{58}$$

In Ref. 6 by a careful analysis of the Bloch waves it is shown that

$$|\hat{\phi}(\mathbf{k},\mathbf{x},u)-\phi(\mathbf{k},\mathbf{x},u)|=O(u),$$
$$|\hat{\varepsilon}(\mathbf{k},u)-\varepsilon(\mathbf{k},u)|=O(u^2), \tag{59}$$

where $\phi(\mathbf{k},\mathbf{x},u)$ are the Bloch waves i.e., the solutions of Eq. (4) and $\varepsilon(\mathbf{k},u)$ the dispersion relation. The fourth statement of the theorem then follows.

Finally, the lower bound on the spectral gap can be obtained by repeating the computations in the preceding sections using an analytic partition of the unity instead of a C^∞ one, see Sec. III. In fact in this way one can prove that the Fourier transform of the two-point Schwinger function is bounded as a function of k_0 in a strip of the imaginary plane of width $\hat{u}(p_F)/2$. We preferred a C^∞ decomposition, as it makes the exposition more readable and intuitive, but there should be no problem to use an analytic decomposition.

At the end, note that our results are independent on the constant $\widetilde{C}$, which is arbitrary; in fact η_1,η_2 as well as the constants entering in the bounds of the theorem do not depend on $\widetilde{C}$, which only affects the convergence radius of the series, i.e., there is an optimal way to choose it; analogue considerations can be made for the parameter γ.

VI. CONCLUSIVE REMARKS

A renormalization-group treatment allows us to clarify this intuition of equivalence between different models in this context. Namely we can consider two models equivalent if their β functions coincide up to terms $O(\gamma^h)$. This means that, see Eq. (48), the critical indices coincides at least at the first order. In this sense we think that the above analysis makes clear that the model with Hamiltonian Eq. (1) in the spinless case and at $p_F=q\pi/a$ is equivalent to the massive Luttinger model Eq. (15); one can repeat for this model the same renormalization-group analysis almost without changes: the only (trivial) differences are that the dispersion relation is linear [so, for instance, the terms quadratic in $\mathbf{k}$ in Eq. (37) are not present] and that $\nu_h=\tau_h=\vartheta_h\equiv 0$ for symmetry reasons. In other words, the relation between the filled-band model Eq. (1) and the massive Luttinger model Eq. (15) is the same of the one[20] between the $u=0$ model Eq. (1) and the Luttinger model.

Another equivalent model is a relativistic quantum-field model, the Yukawa$_2$ model, describing a relativistic massive fermion in $d=1$. One can easily convince himself of this fact noting that the anomalous propagator Eq. (37) can be written as

$$g^{(h)}(x-y)=g_Y^{(h)}(x-y)+C_1^h(x-y), \tag{60}$$

where

$$g_Y^{(h)}(x-y)=\frac{1}{Z_h}\int dk\widetilde{f}_h(k)e^{ik(x-y)}\,\frac{\not{k}+\sigma_h I}{k^2+\sigma_h^2} \tag{61}$$

with $\not{k}=ik_0\gamma_0+2\pi\mathbf{k}\gamma^1$, if γ_0,γ_1 are the γ-matrices for a relativistic $d=1$ Fermi field with light velocity 2π; $g_Y^{(h)}$ is dominant as

$$|C_1^{(h)}(x-y)|\le A\,\frac{\gamma^{2h}}{Z_h}\,\frac{C_N}{1+(\gamma^h|x-y|)^N}$$

with A a suitable constant matrix, i.e., it obeys to a similar bound with an extra factor γ^h; g_Y^h is just the infrared propagator at scale h of the Yukawa$_2$ model (σ_h is just the fermion mass at scale h) and it is trivial to check that the two models are equivalent according to the above definition.

As a conclusion let us discuss briefly the spinning case. The only difference with the spinless case treated here is that there are more running-coupling constants corresponding to quartic monomials in the fields, and the second-order β function is in this case much more complex. One can verify that the dynamical-system correspondent to the second-order β function has a variety of behaviors; it seems that only for special choices of the potential might everything stay unchanged, with respect to the spinless case, while, in general, new ideas seem necessary.

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acting theory, is not unique; this is not too surprising as also in the free $\lambda=0$ case to many values of the chemical potential i.e., $\mu\in\{\varepsilon[(q\pi/a)^+,u],\varepsilon[(q\pi/a)^-,u]\}$ it corresponds a unique value of the Fermi momentum, i.e., $p_F=q\pi/a$.

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