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On a conjecture for the critical behaviour of KAM tori

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Abstract. *At the light of recent results in literature we review a conjecture formulated in* Math. Phys. Electron. J. **1** (1995), paper 5, 1–13, *about the mechanism of breakdown of invariant sets in KAM problems and the identification of the dominant terms in the perturbative expansion of the conjugating function. We show that some arguments developed therein can be carried out further only in some particular directions, so limiting a possible future research program, and that the mechanism of break down of invariant tori has to be more complicated than as conjectured in the quoted paper.*

1. Introduction

1.1. *The state of the art of [GGM].* In [GGM] a conjecture about the mechanism of breakdown of KAM invariant tori is proposed. Roughly it is based on the following idea (we refer to [GGM] for a more detailed and technical exposition and also for the introduction of the notions used in the following analysis).

If $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, let $\mathbb{T}^\ell$ be the ℓ-dimensional torus. Consider a Hamiltonian system

$$\mathcal{H} = \underline{\omega}_0 \cdot \underline{A} + \frac{1}{2}\underline{A} \cdot J^{-1}\underline{A} + \varepsilon f(\underline{\alpha}) , \tag{1.1}$$

where $(\underline{\alpha}, \underline{A}) \in \mathbb{T}^\ell \times \mathbb{R}^\ell$ are conjugate variables, J is the matrix of momenta of inertia, $\cdot$ denotes the inner product in $\mathbb{R}^\ell$, $f(\underline{\alpha})$ is a trigonometric polynomial in the angle variables and ε is a parameter.

For concreteness one can suppose that $\mathcal{H}$ describes the Escande-Doveil pendulum, [ED]: $\ell = 2$ and $f(\underline{\alpha}) = a\cos\alpha_1 + b\cos(\alpha_1 - \alpha_2)$, with $(a, b) \in \mathbb{R}^2$.

The solutions of the equations of motion describing the invariant tori for the system (1.1) with Diophantine rotation vector $\underline{\omega}$ can be parameterized as

$$\underline{\alpha} = \underline{\psi} + \underline{h}(\underline{\psi}; \varepsilon) , \qquad \underline{A} = \underline{A}_0 + \underline{H}(\underline{\psi}; \varepsilon) , \tag{1.2}$$

where $\underline{\psi} \in \mathbb{T}^\ell$, $\underline{A}_0 = J(\underline{\omega} - \underline{\omega}_0)$ and $\underline{h}, \underline{H}$, for ε small enough, are analytic functions, whose series expansions admit a graph representation in terms of trees; we refer to [GGM] for details and definitions.

One can consider also trees in which no resonances (see [GGM], §4) are allowed to appear but the perturbative parameter ε is replaced by $\eta_\varepsilon \equiv \varepsilon(1 - \sigma_\varepsilon)^{-1}$, with a suitable (matrix) *form factor* σ_ε taking into account the resummation of all resonances, [GM], and denote by $\underline{h}^*, \underline{H}^*$ the functions so obtained; the series expansion in η will be called *resummed series.*

If ρ is the radius of convergence of the series defining $\underline{h}, \underline{H}$ in (1.2) and $\varepsilon_c \in \mathbb{R}^+$ is the (positive) critical value at which the tori break down, one has $\varepsilon_c \geq \rho$; in general the analyticity domain in ε of the series defining $\underline{h}, \underline{H}$ is not a circle and it can happen that $\varepsilon_c > \rho$.

One can imagine that the following scenario arises: the singular behaviour of $\underline{h}(\underline{\psi}; \varepsilon), \underline{H}(\underline{\psi}; \varepsilon)$ as functions of ε is the same of that of $\underline{h}^*(\underline{\psi}; \eta_\varepsilon), \underline{H}^*(\underline{\psi}; \eta_\varepsilon)$ and while, for $\varepsilon \to \varepsilon_c^-$, σ_ε is still finite, $\eta = \eta_\varepsilon$ goes out the convergence domain (in η) of the series for $\underline{h}^*, \underline{H}^*$. If moreover the analyticity domain in η of $\underline{h}^*, \underline{H}^*$ turns out to be a circle this would mean that η is a more natural parameter for the invariant tori.

If this really happens then the behaviour of the series near the critical value is determined by the trees without resonances, for which the perturbative series would still be meaningful for ε near ε_c.

Under such an assumption a universal behaviour of the series near the critical value is proposed in [GGM]. Predictions of the value of the critical exponent δ are made, conjecturing that only some simple classes of trees are relevant: the linear trees presenting the largest possible number of small divisors (see [GGM], §5).

Considering (for concreteness purposes) a rotation vector $\underline{\omega} = (r, 1)$, where r is the golden mean (see §3.1 below), then one was led to expect that, by denoting by $\{q_n\}$ the denominators of the convergents defined by the continuous fraction expansion for r, a suitable function $Z(\Lambda_{q_n})$, defined in equation (5.2) of [GGM] and recalled in §3.1 below, satisfied the asymptotics

$$|Z(\Lambda_{q_n})| \approx \frac{C^{q_n}}{q_n^\delta} , \qquad C = C(\eta, f) , \tag{1.3}$$

for some positive constant $C(\eta, f)$ depending on ηf.

1.2. *About the standard and semistandard maps.* If true, the above mechanism should work also for the standard map, which is the dynamical system generated by the iteration of the area-preserving map of the cylinder to itself

$$\begin{cases} x' = x + y + \varepsilon \sin x \ , \\ y' = y + \varepsilon \sin x \ , \end{cases} \tag{1.4}$$

where $(x, y) \in \mathbb{T} \times \mathbb{R}$. At least it is generally assumed that area-preserving maps and Hamiltonian flows share the same critical behaviour: in particular they are expected to have the same critical exponent δ; see also [GGM], [CGJ] and comments therein about [M1], [M2].

The rôle of the functions $\underline{h}, \underline{H}$ is now played by two scalar functions u, v (see, for instance, [BG1]) such that

$$x = \alpha + u(\alpha, \varepsilon) \ , \qquad y = 2\pi\omega + v(\alpha, \varepsilon) \ , \tag{1.5}$$

with ω a fixed rotation number and u, v analytic in their arguments [1]. In terms of $\alpha \in \mathbb{T}$, the dynamics is a trivial rotation: $\alpha \to \alpha' = \alpha + 2\pi\omega$.

Analogously to the Hamiltonian case (1.1), one can define two functions u^*, v^*, obtained by considering only the values of the trees contributing to u, v without resonances (and replacing ε with η, where $\eta \equiv \eta_\varepsilon = \varepsilon(1 - \sigma_\varepsilon)^{-1}$ takes into account the resummation of resonances; note that now σ_ε is a scalar).

For the standard map, let us denote by $\rho(\omega)$ and $\varepsilon_c(\omega)$, respectively, the radius of convergence (in ε) and the critical value for fixed rotation number ω.

Consider also the semistandard map, introduced by Chirikov, [C],

$$\begin{cases} x' = x + y + \varepsilon' \exp ix \ , \\ y' = y + \varepsilon' \exp ix \ , \qquad \varepsilon' = \varepsilon/2 \ , \end{cases} \tag{1.6}$$

and call u_0, v_0 the corresponding conjugating functions; let us denote by $\rho_0(\omega)$ the radius of convergence (in ε') for the semistandard map[2].

1.3. *The implications of [D1].* Given a rotation number $\omega \in (0, 1)$, let $B(\omega)$ be the function

$$B(\omega) = \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} \ , \tag{1.7}$$

where $\{q_n\}$ are the convergents defined by the continued fraction expansion for the rotation number $\omega = [a_1, a_2, a_3, \ldots]$, so that $q_n = a_n q_{n-1} + q_{n-2}$, for $n \geq 1$, and $q_{-1} = 0$, $q_0 = 1$. The function (1.7) is related to the Bryuno function introduced by Yoccoz, [Y].

Consider the series defining $u^* \equiv u^*(\alpha, \eta)$. Of course one can write

$$u^*(\alpha, \eta) = \sum_{k=1}^{\infty} \eta^k \left(\sum_{\nu=-k}^{k} u_\nu^{*(k)} e^{i\nu\alpha} \right) \equiv \sum_{k=1}^{\infty} \eta^k u^{*(k)}(\alpha) \ . \tag{1.8}$$

The radius of convergence $R(\omega)$ of a series like (1.8) can be expressed as

$$R^{-1}(\omega) = \sup_{\alpha \in [0, 2\pi]} \limsup_{k \to \infty} \left| u^{*(k)}(\alpha) \right|^{1/k} = \limsup_{k \to \infty} \max_{|\nu| \leq k} \left| u_\nu^{*(k)} \right|^{1/k} \ , \tag{1.9}$$

as the equivalence between the two definitions has been shown in [D1].

On the other hand the coefficient in (1.8) with $\nu = k$ is the same for both the functions u and u^* (and one has $u_k^{(k)} = u_k^{*(k)} = u_{0k}^{(k)}$). Moreover it is easy to see that one has

$$\left| u_{0k}^{(k)} \right| \geq B_1^k e^{2B(\omega)k} \tag{1.10}$$

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[1] The functions u, v are trivially related: $v(\alpha,\varepsilon) = u(\alpha,\varepsilon) - u(\alpha - 2\pi\omega, \varepsilon)$.

[2] We could define also the functions u_0^*, v_0^* and the critical value $\varepsilon_{0c}(\omega)$ for the semistandard map, but one has $u_0 = u_0^*, v_0 = v_0^*$, as the semistandard map has no resonances, and $\varepsilon_{0c}(\omega) = \rho_0(\omega)$: to deduce the latter property simply note that the semistandard map is invariant under rotation of ε' in the complex plane.

for k large enough and some constant B_1; again see [D1].

This implies that

$$\limsup_{k\to\infty} \max_{|\nu|\le k} \left| u_\nu^{*(k)} \right|^{1/k} \ge B_1 e^{2B(\omega)} , \tag{1.11}$$

so that $R(\omega) \le B_1^{-1} e^{-2B(\omega)}$, *i.e.* the radius of convergence in η of the function u^* at best should be of order of $\rho_0(\omega)$ [3].

1.4. *The implications of [D2].* One has $\rho_0(\omega) = C_1(\omega) e^{-2B(\omega)}$ with $C_1(\omega)$ satisfying the bound $C_1^{-1} < C_1(\omega) < C_1$, uniformly in ω, for a suitable constant C_1, [D1]. Also for the standard map one has the same dependence of the radius of convergence $\rho(\omega)$ on the function $B(\omega)$, with a possibly different function $C_1(\omega)$, always admitting a bound from below and from above: the bound from above is proven in [D1], by using the argument recalled in §1.3, and the one from below in [BG2].

In [D2] it is proven that for rotation numbers [4] $\omega = \gamma_n \equiv [n, n, n, \ldots]$, with n large enough, one has $C_2/n > \varepsilon_c(\gamma_n) > C_2^{-1}/n$, for some constant C_2; as $B(\omega) > \log n$, for $\omega = \gamma_n$, then $\varepsilon_c(\gamma_n) > C_2^{-1} n^{-1}$ and $\rho(\gamma_n) < C_1 n^{-2}$, (see [D2], end of section 5). Therefore one can choose n so large that $\rho(\gamma_n)/\varepsilon_c(\gamma_n)$ is smaller than any prefixed quantity.

Note that the difference between the radius of convergence and the critical value becomes relevant only for some rotation numbers (like the noble numbers γ_n, with n large enough). For instance for γ_1, according to numerical simulations, the analyticity domain appears (very) slightly stretched along the imaginary axis so that not only the two quantities are comparable, but even $\varepsilon_c(\gamma_1) = \rho(\gamma_1)$, [FL].

2. Discussion

2.1. *About Equation (1.3).* If the dominant contributions to u^*, v^* were given by summing only the values of trees defining the functions $Z(\Lambda_{q_n})$, as conjectured in [GGM], then the standard map and the semistandard map should have the same critical behaviour. In fact in terms of trees the semistandard map admits the same graph representation of the standard map, with the only (remarkable) difference that the mode labels have all the same signs; then the paths with the maximal number of small divisors in $Z(\Lambda_{q_n})$ are the same for both the standard map and the semistandard map.

This means that, accepting the above conjecture, one ought to expect that the radius of convergence (in ε) of the functions u_0, v_0 for the semistandard map and the radius of convergence (in η) of the functions u^*, v^* for the standard map should be equal to each other and, in particular, of size of $\rho_0(\omega)$.

Furthermore the results listed in §1.3 imply that the functions u^*, v^* can not have a radius of convergence larger than that of the functions u_0, v_0, also considering all trees contributing to them and not only the ones defining the function $Z(\Lambda_{q_n})$.

In other words the semistandard map should capture all the critical behaviour of the standard map: if so there would be contradiction with the results existing in literature. As a matter of fact we shall see in §3 that the behaviour (1.3) conjectured for $Z(\Lambda_{q_n})$ does not hold.

2.2. *About the behaviour of σ_ε near ε_c.* The idea that the critical behaviour can be studied through the series obtained be neglecting the resonances (that is by resumming them and defining a new parameter $\eta \equiv \eta_\varepsilon$, as briefly recalled in §1.1 and in §1.2) is presented in [GGM]. One could also interpret the numerical results of [CGJ], [CGJK] as supporting such an idea: the fact that the KAM iteration represents the good transformation to look at also far from

[3] In the same way one finds that the radius of convergence in ε of the function u at best should be of order of $\rho_0(\omega)$.

[4] With the notations in §1.1, one has $r=\gamma_1$.

the KAM analyticity domain could suggest that a perturbative approach to the study of the breakdown of KAM invariant tori in some sense should be possible: this was the idea underlying the analysis performed in [GGM]. Anyway much work has still to be done in this direction; see also §4 below.

From the results listed in §1 the following picture emerges. For ε small enough $|\sigma_\varepsilon| < R|\varepsilon|$ for some constant R; see the theorem in [GGM], §4. When ε grows along the real axis toward the critical value, the possibility to have σ_ε still finite for $\varepsilon = \varepsilon_c$ is consistent with a perturbative approach if one of the following two cases arise:
(1) if σ_ε is finite and smooth in ε for $\varepsilon = \varepsilon_c$, then it can happen that $|\eta_c| \equiv |\varepsilon_c(1-\sigma_{\varepsilon_c})^{-1}|$ becomes equal to the radius of convergence $R(\omega)$ of the functions u^*, v^*;
(2) if σ_ε is finite and singular in ε for $\varepsilon = \varepsilon_c$, while $\eta_c = \varepsilon_c(1-\sigma_{\varepsilon_c})^{-1}$ is smaller than the radius of convergence $R(\omega)$ of the functions u^*, v^*, then the singularity of the tori shows up through the singular dependence of σ_ε on ε at the critical value.

In both cases the perturbative series for u^*, v^* can be used also near the critical value, because for any $|\varepsilon| < \varepsilon_c$ one has that η is smaller than the radius of convergence of the series.

In the case of $\omega = \gamma_n$, in principle two possibilities can be envisaged for the behaviour of σ_ε for ε near ε_c:
(i) either $|\sigma_\varepsilon| < 1$,
(ii) or else σ_ε negative and $|\sigma_\varepsilon| \geq 1$ [5].

Consider $\omega = \gamma_n$, with n large enough, and assume the results in [D2]. Then, if the analyticity domain in η was a circle with radius $R(\omega)$ and, for $\varepsilon \to \varepsilon_c^-(\omega)$, $|\sigma_\varepsilon| < 1$ and $\eta \to R(\omega)$, one should have $|\eta| = |\varepsilon(1-\sigma_\varepsilon)^{-1}| \to R(\omega) \approx \rho_0(\omega)$, hence $|1-\sigma_\varepsilon| \to |1-\sigma_{\varepsilon_c}| \approx \varepsilon_c(\omega)/\rho_0(\omega) > C_0 n$, with $C_0^{-1} = C_1 C_2$, which is incompatible with $|\sigma_\varepsilon| < 1$. So only the possibility (ii) above could be consistent both with [D2] and with the scenario proposed in [GGM], for $\omega = \gamma_n$.

More generally, for $\omega = \gamma_n$, with n large enough, if one had $\eta \to \eta_c \leq R(\omega)$ for $\varepsilon \to \varepsilon_c^-(\omega)$, then $|\sigma_\varepsilon|$ should become at least of order $e^{B(\omega)}$ for $\varepsilon \to \varepsilon_c^-(\omega)$, *i.e.* $|\sigma_\varepsilon| \to |\sigma_{\varepsilon_c}| \geq C_3(\omega) e^{B(\omega)}$, for some function $C_3(\omega)$ bounded uniformly in ω: in fact in this way one would have $|\eta_c| \equiv |\varepsilon_c(1-\sigma_{\varepsilon_c})^{-1}| \approx C_3 \varepsilon_c(\omega) e^{-B(\omega)} \approx C_4 \rho_0(\omega) = R(\omega)$, for some constants C_3, C_4, in case (1) and $|\eta_c| < C_4 \rho_0(\omega) = R(\omega)$ in case (2).

3. Analytic results

3.1. *The definition of* $Z(\Lambda_{q_n})$. If $r = (\sqrt{5}-1)/2$ is the *golden mean*, let us call $\{p_n/q_n\}$ the convergents of the continuous fraction expansion for r, where $\{p_n\}$ is the *Fibonacci sequence* defined by $p_{n+1} = p_n + p_{n-1}$ with $p_{-1} = 1$ and $p_0 = 0$, so that $p_n = q_{n+1}$ with $q_{-1} = 0$ and $q_0 = 1$.

The sequence of numbers $Z_n \equiv Z(\Lambda_{q_n})$ is defined in [GGM], equation (5.2), as sum of the values of a suitable class of trees which can be described by the family Λ_{q_n} of self-avoiding walks on $\mathbb{Z}^2$ starting at $(0,0)$, ending at $(q_n, -p_n)$ and contained in the strip $0 < x < q_n$, except for the left extreme points.

Then the numbers Z_n can be approximately defined by the recursive relation

$$Z_{n+1} = Z_n Z_{n-1} \left(\frac{\varepsilon_{n-1}}{\varepsilon_{n+1}} \right)^2 , \tag{3.1}$$

where, for consistency, we fix $Z_{-1} = r^2$ and $Z_0 = r^{-2}$, [GGM]. Set also $\varepsilon_n = q_n r - p_n$; then $\varepsilon_{-1} = -1$ and $\varepsilon_n = q_n r - q_{n-1}$.

[5] Note that σ_ε positive and $|\sigma_\varepsilon| \geq 1$ is not possible as the analyticity of σ_ε in ε for small ε would imply the existence of a value $\bar\varepsilon$ such that $|1-\sigma_{\bar\varepsilon}| = 0$.

Instead of studying (3.1) define $\lambda_n = \log Z_n$, so that

$$\lambda_{n+1} = \lambda_n + \lambda_{n-1} + 2\left(\log \varepsilon_{n-1} - \log \varepsilon_{n+1}\right) \ , \tag{3.2}$$

with $\lambda_{-1} = 2\log r$ and $\lambda_0 = -2\log r$.

3.2. *Against the (1.3).* All the following identities can be easily proven by induction using the fact that r is the positive solution of $r^2 + r - 1 = 0$.

First note that $\varepsilon_n = (-1)^n r^{n+1}$. It follows that

$$\lambda_{n+1} = \lambda_n + \lambda_{n-1} - 4\log r \ . \tag{3.3}$$

which satisfies $\lambda_n = \mu_n + s_n$, where

$$\begin{cases} s_n = -4\log r \sum_{i=0}^{n-1} q_{n-1-i} \ , & \text{for } n \geq 1 \ , \\ s_0 = s_{-1} = 0 \ , \end{cases} \tag{3.4}$$

and μ_n is defined by $\mu_{n+1} = \mu_n + \mu_{n-1}$, with $\mu_{-1} = 2\log r$ and $\mu_0 = -2\log r$. It is now immediate that $\mu_n = -\left(2\log r\right) q_{n-2}$, for $n \geq 1$, so that we have

$$\lambda_n = -2\log r \left(q_{n-2} + 2\sum_{i=0}^{n-1} q_i \right) \ , \tag{3.5}$$

for $n \geq 1$.

Let now use that

$$q_n = \frac{r^{-(n+1)} - (-1)^{n+1} r^{n+1}}{r + r^{-1}} \ . \tag{3.6}$$

Inserting this expression in (3.4) we get, for $n \geq 1$,

$$\begin{aligned} \lambda_n &= \frac{2\log r^{-1}}{r + r^{-1}} \left(r^{-(n-1)} + 2\sum_{i=0}^{n} r^{-i} - (-1)^{n-1} r^{n-1} - 2\sum_{i=0}^{n} (-1)^i r^i \right) = \\ &= 2\log r^{-1} \left(r^{-(n+2)} - 2 + (-1)^{n+2} r^{n+2} \right) \ , \end{aligned} \tag{3.7}$$

which can be written as

$$\lambda_n = 2\log r^{-1} \left[(r + r^{-1}) q_{n+1} - 2 + 2(-1)^{n+2} r^{n+2} \right] \ , \tag{3.8}$$

which means

$$Z_n = r^4 r^{-2(r+r^{-1}) q_{n+1}} r^{-4(-1)^{n+2} r^{n+2}} \ . \tag{3.9}$$

This implies, using that

$$r^{n+1} = \frac{1}{r + r^{-1}} q_n^{-1} \left[1 + (-1)^n r^{2(n+1)} \right] \ , \tag{3.10}$$

the following asymptotics:

$$|Z(\Lambda_{q_n})| \approx K C^{q_{n+1}} \left(1 + (-1)^n d q_{n+1}^{-1} \right) \ , \tag{3.11}$$

with $K = r^{-4}$, $C = r^{-2(r+r^{-1})}$ and $d = 4(r + r^{-1})^{-1} \log r^{-1}$. This is clearly in contradiction with (1.3).

4. Conclusions

4.1. *Dominant contributions.* Our attention to [GGM] has been called back by the recent papers [CGJ], [CGJK], where the breakdown of KAM invariant tori (for two-dimensional Hamiltonian systems) is numerically studied through a renormalization group scheme, and by the results in [D2].

Even if it is not true that the terms defining $Z(\Lambda_{q_n})$ are the only relevant ones (as the results in §3 show), one can argue that the real dominant contributions are given by the trees having the mode labels accumulating near the resonant line (*i.e.* such that the small divisors are really as small as possible), but not necessarily belonging to the class described by Λ_{q_n}.

In fact let us compare the tree values for the Escande-Doveil pendulum with the ones for the standard map and accept that they have the same singular behaviour at the critical value (as it is generally believed, [ED]): the small divisors are respectively $(i\underline{\omega}\cdot\underline{\nu})^2$, $\underline{\nu}=(\nu_1,\nu_2)\in\mathbb{Z}^2$, and $2[\cos(2\pi\omega\nu)-1]$, $\nu\in\mathbb{Z}$. So one can note that they have the same smallness problem, with the only difference that for the Escande-Doveil pendulum one can have also small divisors which are not small at all (when $\underline{\nu}$ is nearly parallel to $\underline{\omega}$), while for the standard map one can approximate the quantity $2[\cos(2\pi\omega\nu)-1]$ with something of the form $(i(\omega\nu_1+\nu_2))^2$, with $\nu_1=\nu$ and ν_2 such that $|\omega\nu_1+\nu_2|\le 1$: in other words one can expect that the tree expansion for the standard map is very similar to that of the Escande-Doveil pendulum, but it does contain only the trees with the momenta directed along the resonant line. Of course this imply neither that in the case of linear trees the most dominant contributions are the ones with the mode labels having all the same signs nor that only the linear trees are relevant. As a matter of fact the analysis performed in §3 shows that making a so restrictive assumption leads to wrong results.

4.2. *Resummed series: smooth form factors.* In conclusion an overall behaviour like (1.3) could be still possible in principle, even if a larger class of trees ought to be taken into account.

Let us consider $\omega=\gamma_n$, with n large enough and use the results in §2.2. If for $\varepsilon\to\varepsilon_c$ one had $|\sigma_\varepsilon|<1$, then we have seen that the analyticity domain in η of u^*,v^* can not be a circle: rather, defining $\eta_c(\omega)$ as the equivalent of ε_c for the functions u^*,v^*, the ratio $\eta_c(\omega)/R(\omega)$ would be of the same order of the ratio $\varepsilon_c(\omega)/\rho(\omega)$, and no simplification could arise in considering the resummed series u^*,v^* instead of the original ones u,v. In other words η would not be a "natural" parameter.

Of course in the case of the standard map, deep cancelation mechanisms should intervene in order to enlarge the analyticity domain along the real axis of the series for u^*,v^* from $R(\omega)\approx\rho_0(\omega)$ to a quantity $\eta_c(\omega)$ of the same order of $\varepsilon_c(\omega)$. Analogous considerations could be made for the functions $\underline{h}^*,\underline{H}^*$ in the case of Hamiltonian flows.

On the contrary if, for $\varepsilon\to\varepsilon_c$, one had case (1), possibility (ii) – see §2.2 –, so that $\sigma_\varepsilon\to C_2(\omega)e^{B(\omega)}$, the convergence domain in η for u^*,v^* could still be a circle. Of course, if this is the case, the study of the factor form σ_ε could turn out to be a very difficult task: in fact it could become a nonperturbative problem, as the full dependence on ε would be required for $\varepsilon\to\varepsilon_c$.

A perturbative analysis could still be possible for some rotation number, for instance for $\omega=\gamma_1$, when the radius of convergence and the critical value are expected to be equal (see comments at the end of §1.4); anyway we have no evidence of this, so we leave it as an open problem.

4.3. *Resummed series: singular form factors.* Suppose now that at the critical value σ_ε is finite and singular. One can consider the resummed series for the form factor σ_ε, by expressing it as series in η, *i.e.* $\sigma_\varepsilon=F(\eta)=F(\varepsilon(1-\sigma_\varepsilon)^{-1})$, where the function F admits a graph representation in terms of trees without resonances and with ε replaced with η_ε. If for $\varepsilon=\varepsilon_c$ one has that η is inside (enough) the domain in which the perturbative series for the function F converges, then one can try to truncate the series to the first orders, so obtaining an (approximate) implicit equation for $\sigma\equiv\sigma_\varepsilon$: the solution should be singular at the value $\varepsilon=\varepsilon_c$. So the possibility of a perturbative study of the breakdown of KAM invariant curves has not to be excluded [6].

Further investigation (also from a numerical point of view) in this direction would be highly

[6] Note that for $\omega{=}\gamma_n$, with n large enough, one can still define the function $F(\eta)$: the problem is that for ε near to the critical value no truncation of the series would be meaningful.

profitable. Also a comparison between the values of the involved quantities for the standard map and the semistandard map would be very enlightening.

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Appendix A1. Numerical analysis

A1.1. *Motivations.* Although the results of §3 are conclusive we report here on some numerical simulations that suggested us those results. We tried to fit λ_n with a slightly more general relation than (1.3), *i.e.* $\lambda_n = k+\delta \log q_n + cq_n$. The problem of this fit is that $q_n \simeq r^{-n}$,so that we have to compute all the involved quantities with a very high accuracy: if not the constant k and the "linear" term δ will be completely lost. We think that the way we did can be of some interest for the reader.

We used the standard Unix command *bc* that is able to execute computation written in a C-like programming language that operates in fixed point notation with number of arbitrary dimension and an arbitrary but prefixed precision. We compute c_i, k_i, δ_i has the best square fit of (3.1) using the value of λ_m with $m = (i-1)100$ to $m = i100$. The only problem is to choose the precision at which the computation are done in such a way to have a desired precision in the value of c_i, k_i, δ_i. This is what we will discuss in the next subsection.

A1.2. *About the precision.* Let p be our chosen precision, *i.e.* we make all operation with p significant digits after the point. This mean that at every elementary operation creates an error $O(10^{-p})$.

If we call θ_i the error due to this round off at step i it's easy to see that the accumulated error on λ_n is $\sum_{i=0}^{n-1} \theta'_i q_{n-1-i} = O(10^{-p} r^n)$. To compute the best quadratic fit we have to solve the equation $\mathbf{A}\vec{\chi} = \vec{v}$, where $\mathbf{A}$ is the symmetric matrix $\mathbf{A} = \sum_{i=0}^{n} \vec{Q}_n \otimes \vec{Q}_n$, with $\vec{Q}_n = (q_n, \log(q_n), 1)$, $\vec{v} = \sum_{i=0}^{n} \vec{Q}_n \lambda_n$ and $\vec{\chi} = (c, \delta, k)$. This mean that $\mathbf{A}_{1,1} \approx r^{-2n}$, while $\mathbf{A}_{1,2}, \mathbf{A}_{1,3} \approx r^{-n}$ and all the other entries $\mathbf{A}_{i,j}$ of $\mathbf{A}$, with $i \geq j$, are of order 1.

Observing that $(\mathbf{A}^{-1})_{i,j} = \mathbf{A}^{i,j} / \det \mathbf{A}$ and that $\det \mathbf{A} \approx r^{-2n} \pm 10^{-p} r^{-2n}$, we get

$$\begin{cases} (\mathbf{A}^{-1})_{1,1} \approx r^{2n} \pm 10^{-p} \; , \\ (\mathbf{A}^{-1})_{1,3}, (\mathbf{A}^{-1})_{1,2} \approx r^n \pm 10^{-p} r^{-n} \; , \\ (\mathbf{A}^{-1})_{2,2} \; , \; (\mathbf{A}^{-1})_{2,3} \; , \; (\mathbf{A}^{-1})_{3,3} \approx 1 \pm 10^{-p} r^{-2n} \; . \end{cases} \tag{4.1}$$

The above estimates with the fact that $\vec{v} \approx (r^{-2n} \pm 10^{-p} r^{-2n}, r^{-n} \pm 10^{-p} r^{-n}, r^{-n} \pm 10^{-p} r^{-n})$ implies an error on χ of order $10^{-p} r^{-2n}$.

A1.3. *Numerical results.* The numerical results that we get are summarized in the next table, which, in our opinion, requires no comments. According to our error analysis, we fixed the precision to $p = 2000 \log_{10} \omega + 20$ in such a way that the reported numbers should be reliable within 20 digits. The agreement with the analytical evaluation reported in §2 is perfect.

n	c	δ	k
100	3.482081477708027334	-.000312407255699678	-1.915302823598781109
200	3.482081477708027334	-.000000000000000000	-1.924847300238413790
⋮	⋮	⋮	⋮
1000	3.482081477708027334	.000000000000000000	-1.924847300238413790

Table 1. Numerical results. The small deviations for i=1 is due to the fact that our fit for λ_n contains q_n instead of q_{n+1} (see §A1.1 for motivations). Values of n of the form n=100i, $i \in \mathbb{N}$, should be meant to be reported in the table, but no change in the results has been observed starting from n=200: so we can say that numerically the limit is reached for such a value of n.

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