computation times. The reason for this success is that the unsplit strategy, which consists of solving

$$\frac{D^{(q)}}{\Delta t}u^{n+1} - \nu\Delta u^{n+1} + \nabla p^{n+1} = S^{n+1} \qquad [55a]$$

$$\nabla \cdot u^{n+1} = 0, \qquad u^{n+1}_{|\partial\Omega} = 0 \qquad [55b]$$

yields a linear system similar to [37], which usually takes far more time to solve than sequentially solving [54a] and [54b]. It is commonly reported in the literature that the ratio of the CPU time for solving [55a]–[55b] to that for solving [54a–c] ranges between 10 to 30.

See also: Compressible Flows: Mathematical Theory; Computational Methods in General Relativity: The Theory; Geophysical Dynamics; Image Processing: Mathematics; Incompressible Euler Equations: Mathematical Theory; Interfaces and Multicomponent Fluids; Magnetohydrodynamics; Newtonian Fluids and Thermohydraulics; Non-Newtonian Fluids; Partial Differential Equations: Some Examples; Variational Methods in Turbulence.

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Fourier Law

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Introduction

In the famous 1822 treatise by Jean Baptiste Joseph Fourier, *Théorie analytique de la chaleur*, the *Discours préliminaire* opens with: "Primary causes are unknown to us; but are subject to simple and constant laws, which may be discovered by observation, the study of them being the subject of natural philosophy. Heat, like gravity, penetrates every substance of the universe, its rays occupy all parts of space. The object of our work is to set forth the mathematical laws which this element obeys. The theory of heat will hereafter form one of the most important branches of general physics." After a brief discussion of rational mechanics, he continues with the sentence: "But whatever may be the range of mechanical theories, they do not apply to the effects of heat. These make up a special order of phenomena, which cannot be explained by the principles of motion and equilibria." Fourier goes on with a thorough description of the phenomenology of heat transport and the derivation of the partial differential equation describing heat transport: the heat equation. A large part of the treatise is

then devoted to solving the heat equation for various geometries and boundary conditions. Fourier's treatise marks the birth of Fourier analysis. After Boltzmann, Gibbs, and Maxwell and the invention of statistical mechanics in the decades after Fourier's work, we believe that Fourier was wrong and that, in principle, heat transport can and should be explained "by the principles of motion and equilibria," that is, within the formalism of statistical mechanics. But well over a century after the foundations of statistical mechanics were laid down, we still lack a mathematically reasonable derivation of Fourier's law from first principles. Fourier's law describes the macroscopic transport properties of heat, that is, energy, in nonequilibrium systems. Similar laws are valid for the transport of other locally conserved quantities, for example, charge, particle density, momentum, etc. We will not discuss these laws here, except to point out that in none of these cases macroscopic transport laws have been derived from microscopic dynamics. As Peierls once put it: "It seems there is no problem in modern physics for which there are on record as many false starts, and as many theories which overlook some essential feature, as in the problem of the thermal conductivity of [electrically] non-conducting crystals."

Macroscopic Law

Consider a macroscopic system characterized at some initial time, say $t=0$, by a nonuniform

temperature profile $T_0(r)$. This temperature profile will generate a heat, that is, energy current $J(r)$. Due to energy conservation and basic thermodynamics:

$$c_v(T)\frac{\partial}{\partial t}T(r,t) = -\nabla \cdot J \qquad [1]$$

where $c_v(T)$ is the specific heat per unit volume. On the other hand, we know that if the temperature profile is uniform, that is, if $T_0(r) \equiv T_0$, there is no current in the system. It is then natural to assume that, for small temperature gradients, the current is given by

$$J(r) = -\kappa(T(r))\nabla T(r) \qquad [2]$$

where $\kappa(T)$ is the conductivity. Here we have assumed that there is no mass flow or other mode of energy transport besides heat conduction (we also ignore, for simplicity, any variations in density or pressure). Equation [2] is normally called as Fourier's law. Putting together eqns [1] and [2], we get the heat equation:

$$c_v(T)\frac{\partial}{\partial t}T(r,t) = \nabla \cdot [\kappa(T)\nabla T] \qquad [3]$$

This equation must be completed with suitable boundary conditions. Let us consider two distinct situations in which the heat equation is observed to hold experimentally with high precision:

1. An isolated macroscopic system, for example, a fluid or solid in a domain Λ surrounded by effectively adiabatic walls. In this case, eqn [3] is to be solved subject to the initial condition $T(r,0) = T_0(r)$ and no heat flux across the boundary of Λ (denoted by $\partial\Lambda$), that is, $n(r) \cdot \nabla T(r) = 0$ if $r \in \partial\Lambda$ with n the normal vector to $\partial\Lambda$ at r. As $t \to \infty$, the system reaches a stationary state characterized by a uniform temperature $\bar{T}$ determined by the constancy of the total energy.
2. A system in contact with heat reservoirs. Each reservoir α fixes the temperature of some portion $(\partial\Lambda)_\alpha$ of the boundary $\partial\Lambda$. The rest of the boundary is insulated. When the system reaches a stationary state (again assuming no matter flow), its temperature will be given by the solution of eqn [3] with the left-hand side set equal to zero,

$$\nabla \cdot \tilde{J}(r) = \nabla \cdot (\kappa\nabla\tilde{T}(r)) = 0 \qquad [4]$$

subject to the boundary condition $\tilde{T}(r) = T_\alpha$ for $r \in (\partial\Lambda)_\alpha$ and no flux across the rest of the boundary.

The simplest geometry for a conducting system is that of a cylindrical slab of height h and cross-sectional area A. It can be either a cylindrical container filled with a fluid or a piece of crystalline solid. In both cases, one keeps the lateral surface of the cylinder insulated. If the top and the bottom of the cylinder are also insulated we are in case (1). If one keeps the top and the bottom in contact with thermostats at temperatures T_h and T_b, respectively, this is (for a fluid) the usual setup for a Benard experiment. To avoid convection, one has to make $T_h > T_b$ or keep $|T_h - T_b|$ small. Assuming uniformity in the direction perpendicular to the vertical x-axis one has, in the stationary state, a temperature profile $\tilde{T}(x)$ with $\tilde{T}(0) = T_b, \tilde{T}(h) = T_h$ and $\kappa(\tilde{T})d\tilde{T}/dx = \text{const.}$ for $x \in (0, h)$.

In deriving the heat equation, we have implicitly assumed that the system is described fully by specifying its temperature $T(r,t)$ everywhere in Λ. What this means on the microscopic level is that we imagine the system to be in local thermal equilibrium (LTE). Heuristically, we might think of the system as being divided up (mentally) into many little cubes, each large enough to contain very many atoms yet small enough on the macroscopic scale to be accurately described, at a specified time t, as a system in equilibrium at temperature $T(r_i, t)$, where r_i is the center of the ith cube. For slow variation in space and time, we can then use a continuous description $T(r, t)$. The theory of the heat equation is very developed and, together with its generalizations, plays a central role in modern analysis. In particular, one can consider more general boundary conditions. Here we are interested in the derivation of eqn [2] from first principles. This clearly presupposes, as a first fundamental step, a precise definition of the concept of LTE and its justification within the law of mechanics.

Empirical Argument

A theory of heat conduction has as a goal the computation of the conductivity $\kappa(T)$ for realistic models, or, at the very least, the derivation of behavior of $\kappa(T)$ as a function of T. The early analysis was based on "kinetic theory." Its application to heat conduction goes back to the works of Clausius, Maxwell, and Boltzmann, who obtained a theoretical expression for the heat conductivity of gases, $\kappa \sim \sqrt{T}$, independent of the gas density. This agrees with experiment (when the density is not too high) and was a major early achievement of the atomic theory of matter.

Heat Conduction in Gases

Clausius and Maxwell used the concept of a "mean free path" λ: the average distance a particle (atom or

molecule) travels between collisions in a gas with particle density ρ. Straightforward analysis gives $\lambda \sim 1/\rho\pi\sigma^2$, where σ an "effective" hard-core diameter of a particle. They considered a gas with temperature gradient in the x-direction and assumed that the gas is (approximately) in local equilibrium with density ρ and temperature $T(x)$. Between collisions, a particle moves a distance λ carrying a kinetic energy proportional to $T(x)$ from x to $x + \lambda/\sqrt{3}$, while in the opposite direction the amount carried is proportional to $T(x + \lambda\sqrt{3})$. Taking into account the fact that the speed is proportional to $\sqrt{T}$ the amount of energy J transported per unit area and time across a plane perpendicular to the x-axis is approximately

$$J \sim \rho\sqrt{T}\Big[T(x) - T(x + \lambda\sqrt{3})\Big]$$
$$\sim -\sigma^{-2}\sqrt{T}\frac{dT}{dx} \qquad [5]$$

and so $\kappa \sim \sqrt{T}$ independent of ρ, in agreement with experiment. It was clear to the founding fathers that starting with a local equilibrium situation the process described above will produce, as time goes on, a deviation from LTE. They reasoned, however, that this deviation from local equilibrium will be small when $(\lambda/T)dT/dx \ll 1$, the regime in which Fourier's law is expected to hold, and the above calculation should yield, up to some factor of order unity, the right heat conductivity. To have a more precise theory, one can describe the state of the gas through the probability distribution $f(r,p,t)$ of finding a particle in the volume element $dr\,dp$ around the phase space point (r,p). Here LTE means that

$$f(r,p,t) \simeq \exp\left(-\frac{p^2}{2mkT(r)}\right)$$

where m is the mass of the particles. If one computes the heat flux at a point r by averaging the microscopic energy current at r, $j = \rho v(1/2mv^2)$, over $f(r,p,t)$ then it is only the deviation from local equilibrium which makes a contribution. The result however is essentially the same as eqn [5]. This was shown by Boltzmann, who derived an accurate formula for κ in gases by using the Boltzmann equation. If one takes κ from experiment, the above analysis yields a value for σ, the effective size of an atom or molecule, which turns out to be close to other determinations of the characteristic size of an atom. This gave an evidence for the reality of atoms and the molecular theory of heat.

Heat Conduction in Insulating Crystals

In (electrically) conducting solids, heat is mainly transported by the conduction electron. In this case, one can adapt the theory discussed in the previous section. In (electrically) insulating solids, on the other hand, heat is transmitted through the vibrations of the lattice. In order to use the concepts of kinetic theory, it is useful to picture a solid as a gas of phonons which can store and transmit heat. A perfectly harmonic crystal, due to the fact that phonons do not interact, has an infinite thermal conductivity: in the language of kinetic theory, the mean free path λ is infinite. In a real crystal, the anharmonic forces produce interactions between the phonons and therefore a finite mean free path. Another source of finite thermal conductivity may be the lattice imperfections and impurities which scatter the phonons. Debye devised a kind of kinetic theory for phonons in order to describe thermal conductivity. One assumes that a small gradient of temperature is imposed and that the collisions between phonons maintain local equilibrium. An elementary argument gives a thermal conductivity analogous to eqn [5] obtained in the last subsection for gases (remembering, however, that the density of phonons is itself a function of T)

$$\kappa \sim c_v c^2 \tau \qquad [6]$$

where, with respect to eqn [5], ρ has been replaced by c_v, the specific heat of phonons, $\sqrt{T}$ by c, the (mean) velocity of the phonons, and λ by $c\tau$, where τ is the effective mean free time between phonon collisions. The thermal conductivity depends on the temperature via τ, and a more refined theory is needed to account for this dependence. This was done by Peierls via a Boltzmann equation for the phonons. In collisions among phonons, the momentum of phonons is conserved only modulo a vector of the reciprocal lattice. One calls "normal processes" those where the phonon momentum is conserved and "Umklap processes" those where the initial and final momenta differ by a nonzero reciprocal lattice vector. Peierls' theory may be summarized (very roughly) as follows: in the absence of Umklap processes, the mean free path, and thus the thermal conductivity of an insulating solid, is infinite. A success of Peierls' theory is to describe correctly the temperature dependence of the thermal conductivity. Furthermore, on the basis of this theory, one does not expect a finite thermal conductivity in one-dimensional monoatomic lattices with pair interactions. This seems so far to be a correct prediction, at least in the numerous numerical results performed on various models.

Statistical Mechanics Paradigm: Rigorous Analysis

In a rigorous approach to the above arguments, we have to first formulate precisely the problem on a

mathematical level. It is natural to adapt the standard formalism of statistical mechanics to our situation. To this end, we assume that our system is described by the positions Q and momenta P of a (very large) number of particles, N, with $Q = (q_1, \ldots, q_N) \in \Lambda^N, \Lambda \subset \mathbb{R}^d$, and $P = (p_1, \ldots, p_N) \in \mathbb{R}^{dN}$. The dynamics (in the bulk) is given by a Hamiltonian function $H(Q, P)$. A state of the system is a probability measure $\mu(P, Q)$ on phase space. As usual in statistical mechanics, the value of an observable $f(P, Q)$ will be given by the expected value of f with respect to the measure μ. In the case of a fluid contained in a region Λ, we can assume that the Hamiltonian has the form

$$H(P, Q) = \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + \sum_{j \neq i} \phi(q_j - q_i) + u(q_i) \right]$$
$$= \sum_{i=1}^{N} \frac{p_i^2}{2m} + \mathcal{V}(Q) \qquad [7]$$

where $\phi(q)$ is some short-range interparticle potential and $u(q_i)$ an external potential (e.g., the interaction of the particle with fixed obstacles such as a conduction electron interacting with the fixed crystalline ions). If we want to describe the case in which the temperature at the boundary is kept different in different regions $\partial\Lambda_\alpha$, we have to properly define the dynamics at the boundary of the system. A possibility is to use "Maxwell boundary conditions": when a particle hits the wall in $\partial\Lambda_\alpha$, it gets reflected and re-emerges with a distribution of velocities

$$f_\alpha(dv) = \frac{m^2}{2\pi(kT_\alpha)^2} |v_x| \exp\left[-\frac{mv^2}{2kT_\alpha} \right] dv \qquad [8]$$

Several other ways to impose boundary conditions have been considered in the literature. The notion of LTE can be made precise here in the so-called hydrodynamic scaling limit (HSL), where the ratio of microscopic to macroscopic scales goes to zero. The macroscopic coordinates r and t are related to the microscopic ones q and τ, by $r = \epsilon q$ and $t = \epsilon^\alpha \tau$, that is, if Λ is a cube of macroscopic sides l, then its sides, now measured in microscopic length units, are of length $L = \epsilon^{-1} l$. We then suppose that at $t = 0$ our system of $N = \rho L^d$ particles is described by an equilibrium Gibbs measure with a temperature $T(r) = T(\epsilon q)$: roughly speaking, the phase-space ensemble density has the form

$$\mu_0(P, Q) \sim \exp\left\{ -\sum_{i=1}^{N} \beta_0(\epsilon q_i) \right.$$
$$\left. \times \left[\frac{p_i^2}{2m} + \sum_{j \neq i} \phi(q_j - q_i) + u(q_i) \right] \right\} \qquad [9]$$

where $\beta_0^{-1}(r) = T_0(r)$. In the limit $\epsilon \to 0, \rho$ fixed, the system at $t = 0$ will be macroscopically in LTE with a local temperature $T_0(r)$ (as already noted, here we suppress the variation in the particle density $n(r)$). We are interested in the behavior of a macroscopic system, for which $\epsilon \ll 1$, at macroscopic times $t \geq 0$, corresponding to microscopic times $\tau = \epsilon^{-\alpha} t, \alpha = 2$ for heat conduction or other diffusive behavior. The implicit assumption then made in the macroscopic description given earlier is that, since the variations in $T_0(r)$ are of order ϵ on a microscopic scale, then for $\epsilon \ll 1$, the system will, also at time t, be in a state very close to LTE, with a temperature $T(r, t)$ that evolves in time according to Fourier's law, eqn [1]. From a mathematical point of view, the difficult problem is to prove that the system stays in LTE for $t > 0$ when the dynamics are given by a Hamiltonian time evolution. This requires proving that the macroscopic system has some very strong ergodic properties, for example, that the only time-invariant measures locally absolutely continuous with respect to the Lebesgue measure are, for infinitely extended spatially uniform systems, of the Gibbs type. This has only been proved so far for systems evolving via stochastic dynamics (e.g., interacting Brownian particles or lattice gases). For such stochastic systems, one can sometimes prove the hydrodynamical limit and derive macroscopic transport equations for the particle or energy density and thus verify the validity of Fourier law. Another possibility, as we already saw, is to use the Boltzmann equation. Using ideas of hydrodynamical space and time scaling described earlier, it is possible to derive a controlled expansion for the solution of the stationary Boltzmann equation describing the steady state of a gas coupled to temperature reservoirs at the top and bottom. One then shows that for $\epsilon \ll 1$, ϵ being now the ratio λ/L, the Boltzmann equation for f in the slab has a time-independent solution which is close to a local Maxwellian, corresponding to LTE (apart from boundary layer terms) with a local temperature and density given by the solution of the Navier–Stokes equations which incorporates Fourier's law as expressed in eqn [2]. The main mathematical problem is in controlling the remainder in an asymptotic expansion of f in power of ϵ. This requires that the macroscopic temperature gradient, that is, $|T_1 - T_2|/h$, where $h = \epsilon L$ is the thickness of the slab on the macroscopic scale, be small. Even if this apparently technical problem could be overcome, we would still be left with the question of justifying the Boltzmann equation for such steady states and, of course, it would not tell us anything

about dense fluids or crystals. In fact, the Boltzmann equation itself is really closer to a macroscopic than to a microscopic description. It is obtained in a well-defined kinetic scaling limit in which, in addition to rescaling space and time, the particle density goes to zero, that is, $\lambda \gg \sigma$.

A simplified model of a crystal is characterized by the fact that all atoms oscillate around given equilibrium positions. The equilibrium positions can be thought of as the points of a regular lattice in $\mathbb{R}^d$, say $\mathbb{Z}^d$. Although $d = 3$ is the physical situation, one can also be interested in the case $d = 1, 2$. In this situation, $\Lambda \subset \mathbb{Z}^d$ with cardinality N, and each atom is identified by its position $x_i = i + q_i$, where $i \in \Lambda$ and $q_i \in R^d$ is the displacement of the particle at lattice site i from this equilibrium position. Since interatomic forces in real solids have short range, it is reasonable to assume that the atoms interact only with their nearest neighbors via a potential that depends only on the relative distance with respect to the equilibrium distance. Accordingly, the Hamiltonians that we consider have the general form

$$H(P, Q) = \sum_{i \in \Lambda} \frac{p_i^2}{2m} + \sum_{|i-j|=1} V(q_i - q_j) + \sum_i U_i(q_i)$$

$$= \sum_{i \in \Lambda} \frac{p_i^2}{2m} + \mathcal{V}(Q) \qquad [10]$$

where $P = (p_i)_{i \in \Lambda}$ and analogously for Q. We shall further assume that as $|q| \to \infty$ so do $U_i(q)$ and $V(q)$. The addition of $U_i(q)$ pins down the crystal and ensures that $\exp[-\beta H(P, Q)]$ is integrable with respect to $dPdQ$, and thus the corresponding Gibbs measure is well defined. In this case, in order to fix the temperature at the boundary, one can add a Langevin term to the equation of particles on the boundaries, that is, if $i \in \partial \Lambda_\alpha$ the equation for the particle is

$$\dot{p}_i = -\partial_{q_i} H(P, Q) - \lambda p_i + \sqrt{\lambda T_\alpha} \dot{w}_i \qquad [11]$$

where $\dot{w}_i$ is a standard white noise. Other thermostatting mechanisms can be considered. In this case we can also define LTE using eqn [9] but we run into the same difficulties described above – although the problem is somehow simpler due to the presence of the lattice structure and the fact that the particles oscillate close to their equilibrium points. We can obtain Fourier's law only by adding stochastic terms, for example, terms like eqn [11], to the equation of motion of every particle and assuming that $U(q)$ and $V(q)$ are harmonic. These added noises can be thought of as an effective description of the chaotic motion generated by the anharmonic terms in $U(q)$ and $V(q)$.

Just how far we are from establishing rigorously the Fourier law is clear from our very limited mathematical understanding of the stationary nonequilibrium state (SNS) of mechanical systems whose ends are, as in the example of the Benard problem, kept at fixed temperatures T_1 and T_2. Various models have been considered, for example, models with Hamiltonian [10] coupled at the boundaries with heat reservoirs described by eqns [11]. The best mathematical results one can prove are: the existence and uniqueness of SNS; the existence of a stationary nontrivial heat flow; properties of the fluctuations of the heat flow in the SNS; the central-limit theorem type fluctuations (related to Kubo formula and Onsager relations; and large-deviation type fluctuations related to the Gallavotti–Cohen fluctuation theorem). What is missing is information on how the relevant quantities depend on the size of the system, N. In this context, the heat conductivity can be defined precisely without invoking LTE. To do this, we let $\bar{J}$ be the expectation value in the SNS of the energy or heat current flowing from reservoir 1 to reservoir 2. We then define the conductivity κ_L as $\bar{J}/(A\delta T/L)$, where $\delta T/L = (T_1 - T_2)/L$ is the effective temperature gradient for a cylinder of microscopic length L and uniform cross section A, and $\kappa(T)$ is the limit of κ_L when $\delta T \to 0 (T_1 = T_2 = T)$ and $L \to \infty$. The existence of such a limit with κ positive and finite is what one would like to prove.

See also: Dynamical Systems and Thermodynamics; Ergodic Theory; Interacting Particle Systems and Hydrodynamic Equations; Kinetic Equations; Nonequilibrium Statistical Mechanics: Dynamical Systems Approach; Nonequilibrium Statistical Mechanics: Interaction Between Theory and Numerical Simulations.

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Fourier–Mukai Transform in String Theory

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Introduction

The Fourier–Mukai transform has been introduced in the study of abelian varieties by Mukai and can be thought of as a nontrivial algebro-geometric analog of the Fourier transform. Since its original introduction, the Fourier–Mukai transform turned out to be a useful tool for studying various aspects of sheaves on varieties and their moduli spaces, and as a natural consequence, to learn about the varieties themselves. Various links between geometry and derived categories have been uncovered; for instance, Bondal and Orlov proved that Fano varieties, and certain varieties of general type, can be reconstructed from their derived categories. Moreover, Orlov proved a derived version of the Torelli theorem for K3 surfaces and also a structure theorem for derived categories of abelian varieties. Later, Kawamata gave evidence to the conjecture that two birational smooth projective varieties with trivial canonical sheaves have equivalent derived categories, which has been proved by Bridgeland in dimension 3.

The Fourier–Mukai transform also enters into string theory. The most prominent example is Kontsevich's homological mirror-symmetry conjecture. The conjecture predicts (for mirror dual pairs of Calabi–Yau manifolds) an equivalence between the bounded derived category of coherent sheaves and the Fukaya category. The conjecture implies a correspondence between certain self-equivalences (given by Fourier–Mukai transforms) of the derived category and symplectic self-equivalences of the mirror manifold.

Besides their importance for geometrical aspects of mirror symmetry, the Fourier–Mukai transforms have also been important for heterotic string compactifications. The motivation for this came from the conjectured correspondence between the heterotic string and F-theory, which both rely on elliptically fibered Calabi–Yau manifolds. To give evidence for this correspondence, an explicit description of stable holomorphic vector bundles was necessary and inspired a series of publications by Friedman, Morgan, and Witten. Their bundle construction relies on two geometrical objects: a hypersurface in the Calabi–Yau manifold together with a line bundle on it; more precisely, they construct vector bundles using a relative Fourier–Mukai transform.

Various aspects and refinements of this construction have been studied by now. For instance, a physical way to understand the bundle construction can be given using the fact that holomorphic vector bundles can be viewed as D-branes and that D-branes can be mapped under T-duality to new D-branes (of different dimensions).

We survey aspects of the Fourier–Mukai transform, its relative version and outline the bundle construction of Friedman, Morgan, and Witten. The construction has led to many new insights, for instance, the presence of 5-branes in heterotic string vacua has been understood. The construction also inspired a tremendous amount of work towards a heterotic string phenomenology on elliptic Calabi–Yau manifolds. For the many topics omitted the reader should consult the "Further reading" section.

The Fourier–Mukai Transforms

Every object E of the derived category on the product $X \times Y$ of two smooth algebraic varieties X and Y gives rise to a functor Φ^E from the bounded derived category $D(X)$ of coherent sheaves on X to the similar category on Y:

$$\Phi^E : D(X) \rightarrow D(Y)$$

$$F \mapsto \Phi^E(F) = R\hat{\pi}_*(\pi^* F \otimes E)$$

where $\pi, \hat{\pi}$ are the projections from $X \times Y$ to X and Y, respectively, and $\otimes$ denotes the derived tensor product. $\Phi^E(F)$ is called Fourier–Mukai transform with kernel $E \in D(X \times Y)$ (in analogy