A fluctuation theorem for non-equilibrium relaxational systems driven by external forces

Francesco Zamponi[1,2], Federico Bonetto[3], Leticia F Cugliandolo[4,5] and Jorge Kurchan[6]

[1] Dipartimento di Fisica, Università di Roma La Sapienza, Piazzale Aldo Moro 2, 00185 Roma, Italy

[2] SOFT–INFM–CNR, Università di Roma La Sapienza, Piazzale Aldo Moro 2, 00185 Roma, Italy

[3] School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

[4] Laboratoire de Physique Théorique et Hautes Énergies Jussieu, 4 Place Jussieu, 75252 Paris Cedex 05, France

[5] Laboratoire de Physique Théorique, École Normale Supérieure, 24 Rue Lhomond, 75231 Paris Cedex 05, France

[6] PMMH École Supérieure de Physique et Chimie Industrielles, 10 Rue Vauquelin, 75231 Paris Cedex 05, France

E-mail: francesco.zamponi@phys.uniroma1.it, bonetto@math.gatech.edu, leticia@lpt.ens.fr and jorge@pmmh.espci.fr

Abstract. We discuss an extension of the fluctuation theorem to stochastic models that, in the limit of zero external drive, are not able to equilibrate with their environment, extending earlier results of Sellitto. We show that if the entropy production rate is suitably defined, its probability distribution function verifies the fluctuation relation with the ambient temperature replaced by a (frequency dependent) effective temperature. We derive modified Green–Kubo relations. We illustrate these results with the simple example of an oscillator coupled to a non-equilibrium bath driven by an external force. We discuss the relevance of our results for driven glasses and the diffusion of Brownian particles in out-of-equilibrium media and propose a concrete experimental strategy for measuring the low frequency value of the effective temperature using the

fluctuations of the work done by an ac conservative field. We compare our results to related ones that appeared in the literature recently.

Keywords: driven diffusive systems (theory), slow dynamics and ageing (theory), Brownian motion, fluctuations (theory)

ArXiv ePrint: cond-mat/0504750

Contents

1. Introduction

Relatively few generic results for non-equilibrium systems exist. Recently, two such results that apply to seemingly very different physical situations have been proposed and extensively studied. One is the *fluctuation theorem* that characterizes the fluctuations of the entropy production over long time intervals in certain driven steady states [1, 2]. Another one is the extension of the fluctuation-dissipation theorem that relates induced and spontaneous fluctuations in equilibrium to the non-equilibrium slow relaxation of glassy systems [3, 4]. While the former result has been proven for reversible hyperbolic dynamical systems [2, 5] and for the driven stochastic dynamic evolution of an open system coupled to an external environment [6, 7], the latter has only been obtained in a number of solvable mean field models and numerically in some more realistic glassy systems (see [8] for a review). The modification of the fluctuation-dissipation theorem can be rationalized in terms of the generation of an *effective temperature* [9, 10]. The expected thermodynamic properties of the effective temperature have been demonstrated in a number of cases [11][7].

    One may naturally wonder whether these two quite generic results may be included in a common, more general statement. The scope of this article is to discuss this possibility

---

[7] The definition of effective temperature that we use here does not have good thermodynamic properties for all possible non-equilibrium systems. Even if it is still not completely established which are the precise requirements needed to ensure the thermodynamic nature of this temperature, it seems to be clear that the underlying system must have a bounded energy density and that the relaxing dynamics should be slow (a limit of small entropy production, as described in [10]). Some cases where these requirements fail have been discussed in Fielding *et al* [11].

in general, illustrating it with a very simple example in which one can very easily reach the 'driven limit' and the 'non-equilibrium relaxational' case. This project was pioneered by Sellitto [12] who asked the same question some years ago and tried to give an answer to it using a stochastic lattice gas with reversible kinetic constraints in diffusive contact with two particle reservoirs at different chemical potentials. Other developments in similar directions have been proposed and analysed by several authors [13]–[15]. We shall discuss them in section 8.

Before entering into the details of our calculations let us start by reviewing the precise statement of the usual fluctuation theorem and the extension of the fluctuation-dissipation theorem as well as the definition of the effective temperature.

1.1. The fluctuation theorem

The *fluctuation theorem* concerns the fluctuations of the entropy production rate[8], that we call $\sigma(x)$, in the stationary non-equilibrium state of a dynamical system defined by a state variable $x \in M$, where M is the phase space of the system.

In non-equilibrium stationary states the function $\sigma(x)$ has a positive average, $\sigma_+ = \int_M \mathrm{d}x\, \mu_+(x)\sigma(x) > 0$, where $\mu_+(x)$ is the stationary state distribution. This allows one to define

$$p(x) \equiv \frac{1}{\tau\sigma_+}\mathcal{S}_\tau = \frac{1}{\tau\sigma_+}\int_{-\tau/2}^{\tau/2}\mathrm{d}t\ \sigma(S_t x), \tag{1}$$

where $S_t x$, with time $t \in [-\tau/2, \tau/2]$, is a segment of the system's trajectory starting at the point x at $t = -\tau/2$. The *large deviation function* of $p(x)$ is

$$\zeta(p) = \lim_{\tau \to \infty} \tau^{-1} \ln \pi_\tau(p), \tag{2}$$

where $\pi_\tau(p)$ is the distribution of p in the stationary non-equilibrium state. The fluctuation theorem is the following statement about the large deviation function of $p(x)$:

$$\zeta(p) - \zeta(-p) = p\sigma_+. \tag{3}$$

The relation (3) was first discovered in a numerical simulation [1], and subsequently stated as a theorem for reversible hyperbolic dynamical systems (the *Gallavotti–Cohen fluctuation theorem*) [2,5]. The proof was extended to Langevin systems coupled to a white bath in [6] and to generic Markov processes in [7]. Equation (3) was successfully tested in a wide number of numerical simulations—see e.g. [16]–[19]—and, more recently, it was also analysed in experiments on granular materials and turbulent flows [20]–[23]. At present, it is believed to yield a very general characterization of the fluctuations of the entropy production in a variety of out-of-equilibrium stationary states.

In all the cases cited above an external drive maintains the systems in a stationary non-equilibrium regime. In the absence of the drive the systems so far analysed easily equilibrate. In the stochastic case, the systems are in contact with an equilibrated environment at a well defined temperature.

---

[8] We shall not address in this paper the problem of the identification of the phase space contraction rate with the entropy production rate in dynamical systems.

1.2. The effective temperature

The analytic solution to the relaxation of mean field glassy models following a quench into their glassy phase demonstrated that their relaxation occurs out of equilibrium [3, 4]. The reason that these models do not reach equilibrium when relaxing from a random initial condition is that their equilibration time diverges with their size. Thus, when the thermodynamic limit is taken at the outset of the calculation, all times considered are finite with respect to the size of the system. These systems approach a slow non-equilibrium regime in which one observes a breakdown of stationarity and, more importantly for the subject of this paper, a violation of the fluctuation-dissipation theorem that relates spontaneous and induced fluctuations in thermal equilibrium.

The relation between spontaneous and induced fluctuations found in mean field glassy models is, however, surprisingly simple. Let us define the linear response of a generic observable O measured at time t to an infinitesimal perturbation constantly applied since a previous 'waiting time' t_w, and the correlation between the same (unperturbed) observable measured at t_w and t,

$$\chi(t, t_\mathrm{w}) \equiv \int_{t_\mathrm{w}}^{t} \mathrm{d}t' \, R(t, t') = \int_{t_\mathrm{w}}^{t} \mathrm{d}t' \, \left. \frac{\delta \langle O(t) \rangle}{\delta h(t')} \right|_{h=0}, \tag{4}$$

$$C(t, t_\mathrm{w}) \equiv \langle O(t) O(t_\mathrm{w}) \rangle, \tag{5}$$

where, for simplicity, we assumed that the observable O has a vanishing average, $\langle O(t) \rangle = 0$ for all t. In the cases we shall be interested in, the relation between these two quantities takes the form

$$\lim_{t_\mathrm{w} \gg t_0} \chi(t, t_\mathrm{w}) = f[C(t, t_\mathrm{w})] \tag{6}$$

in the limit of long waiting time after the initial time t_0. This equation means that the waiting time and total time dependence in χ enter only through the value of the associated correlation between these times. This is trivially true in equilibrium since the *fluctuation-dissipation theorem* states that

$$f(C) = \frac{1}{T} (1 - C), \tag{7}$$

for all times $t \geq t_\mathrm{w} \geq t_\mathrm{eq}$, where T is the temperature of the thermal bath (and that of the system as well) and we set the Boltzmann constant to one, $k_\mathrm{B} = 1$. Out of equilibrium, $f(C)$ can take a different form. In a large variety of models with *slow dynamics* that describe aspects of glasses, $f(C)$ is a broken line:

$$f(C) = \frac{1}{T} (1 - C) \, \theta(C - q_\mathrm{ea}) + \left[\frac{1}{T} (1 - q_\mathrm{ea}) + \frac{1}{T_\mathrm{eff}} (q_\mathrm{ea} - C) \right] \theta(q_\mathrm{ea} - C); \tag{8}$$

see [8] for a review. This broken line has two slopes, $-1/T$ for $C > q_\mathrm{ea}$ (i.e., small $t - t_\mathrm{w}$) and $-1/T_\mathrm{eff}$ for $C < q_\mathrm{ea}$ (i.e., large $t - t_\mathrm{w}$). The breaking point q_ea has an interpretation in terms of intra-cage and out-of-cage motion in the relaxation of structural glasses, inter-domain and domain wall motion in coarsening systems etc. Since T_eff is found to be larger than T, the second term violates the fluctuation-dissipation theorem. We have used the suggestive name *effective temperature*, T_eff, to parametrize the second slope. The justification is that for mean field glassy models—and within all resummation schemes

applied to realistic ones as well—T_{eff} does indeed behave as a temperature [9]. A similar relation was later found numerically in the slow relaxational dynamics of a number of more realistic glassy systems such as Lennard-Jones mixtures [24]. More complicated forms, with a sequence of segments with different slopes, appear in mean field glassy models of the Sherrington–Kirkpatrick type.

The general definition of the effective temperature is

$$-\frac{1}{T_{\text{eff}}(C)} = \frac{\mathrm{d}\chi}{\mathrm{d}C}. \tag{9}$$

In order to be consistent with the thermodynamic properties one needs to find a single value of T_{eff} in each time regime as defined by the correlation scales of [4].

So far we considered systems relaxing out of equilibrium *in the absence of an external drive*. The extension of the definition of the effective temperature to the *driven* dynamics of glasses and supercooled liquids was again motivated by the analytic solution of mean field models under non-potential forces [25]–[27]. The existence of non-trivial effective temperatures, i.e. of functions $f(C)$ that differ from equation (7), was also observed in numerical simulations of sheared Lennard-Jones mixtures [28, 29] and in a number of other driven low dimensional models [30].

In the case of relaxing glasses the dynamics occurs out of equilibrium because below some temperature the equilibration time falls beyond all experimentally accessible timescales. These macroscopic systems then evolve out of equilibrium even if they are in contact with a thermal reservoir, itself in equilibrium at a given temperature T. Under the effect of stirring forces, supercooled liquids and glasses are typically driven into a non-equilibrium stationary regime, even for relatively modest flow.

1.3. A connection between the two?

The fluctuation theorem and the fluctuation-dissipation theorem are related: indeed, it was proven that, for systems which are able to equilibrate in the small entropy production limit ($\sigma_+ \to 0$), the fluctuation theorem implies the Green–Kubo relations for transport coefficients, that are a particular instance of the fluctuation-dissipation theorem [6, 31, 32].

It is now natural to wonder what the fate of the fluctuation theorem is if the system on which the driving force is applied cannot equilibrate with its environment and evolves out of equilibrium even in the absence of the external drive. Our aim is to investigate whether the fluctuation theorem is modified and, more precisely, whether the effective temperature [9] enters its modified version. In particular, this question will arise if the limit of large sampling time, τ in equations (1) and (2), is taken after the limit of large system size. The order of the limits is important because a finite size undriven system will always equilibrate with the thermal bath in a large enough time τ. As the fluctuation theorem concerns the fluctuations of σ for $\tau \to \infty$, if one wants to observe *intrinsic non-equilibrium* effects, the latter limit has to be taken *after* the thermodynamic limit. Some conjectures about this issue have recently appeared in the literature [12]–[15] and we discuss them in section 8.

Our idea is to study the relaxational and driven dynamics of simple systems such that the effective temperature is not trivially equal to the ambient temperature. For a system coupled to a single thermal bath, this happens whenever:

(i) the thermal bath has temperature T, but the system is not able to equilibrate with the bath—this is realized by the glassy cases discussed above, provided that the sampling time is smaller than the equilibration time; and/or

(ii) the system is very simple (not glassy) but it is set in contact with a bath that is not in equilibrium—one can think of two ways of realizing this: one is with a single bath represented by a thermal noise and a memory friction kernel that do not verify the fluctuation-dissipation relation [33] (a situation that arises if one considers the diffusion of a Brownian particle in a complex medium, e.g. a glass, or granular matter [34]–[36]), and in this case the medium, which acts as a thermal bath with respect to the Brownian particle, is itself out of equilibrium; another example is that of a system coupled to a number of equilibrated thermal baths with different timescales and at different temperatures [10].

1.4. Effective equations for the dynamics of mean field glasses

The cases (i) and (ii) mentioned in the previous section are closely related as, at least at the mean field level, the problem of glassy dynamics can be mapped onto the problem of a single 'effective' degree of freedom moving in an out-of-equilibrium self-consistent environment [10, 33]. Situations (i) and (ii) are then described by the same kind of equation, namely, a Langevin equation for a single degree of freedom coupled to a non-equilibrium bath.

Indeed, in the study of mean field models for glassy dynamics [10, 33] and when using resummation techniques within a perturbative approach to microscopic glassy models with no disorder, it is possible to reduce the N-dimensional equations of motion to a single equation for an 'effective' variable by means of a saddle-point evaluation of the dynamic generating functional. These equations are valid in the large size $N \to \infty$ limit; the discussion that follows concerns the fluctuation relation involving $\mathcal{S}_\tau$ for $\tau \to \infty$ taken *after* $N \to \infty$: for these finite timescales τ a fluctuation relation involving effective temperatures may and will arise, while for the extreme times $\tau \to \infty$ before $N \to \infty$ the usual fluctuation theorem (involving only the bath temperature T) holds.

The effective equation of motion of a single spin at finite time reads

$$\gamma \dot\varphi_t = -\mu(t)\,\varphi_t + \int_{-\infty}^{t} \mathrm{d}t'\,\Sigma(t,t')\,\varphi_{t'} + \rho_t + \alpha h[\varphi_t], \qquad (10)$$

where ρ_t is a Gaussian noise such that $\langle \rho_t \rho_{t'} \rangle = D(t,t')$. The 'self-energy' $\Sigma(t,t')$ and the 'vertex' $D(t,t')$ depend on the interactions in the system. They cannot be calculated exactly for generic interactions but they can be approximated within different resummation schemes (mode coupling, self-consistent screening etc) or calculated explicitly for disordered mean field models. The last term (if present) represents an external drive.

In the absence of the driving force equation (10) shows a *dynamical transition* at a temperature $T_{\rm d}$. Above $T_{\rm d}$, in the limit $t, t' \to \infty$ the functions $\Sigma(t,t')$ and $D(t,t')$ become time-translation invariant and are related by the fluctuation-dissipation relation with Σ playing the role of a response and D being a correlation. Below $T_{\rm d}$ the system is no longer able to equilibrate with the thermal bath and *ages* indefinitely, i.e. the functions $\Sigma(t,t')$ and $D(t,t')$ depend on t and t' separately even in the infinite time limit, and

the relaxation time τ_α grows indefinitely. At low temperatures the fluctuation-dissipation theorem does not necessarily hold, and the relation between Σ and D can be used to measure the effective temperature for the particular problem at hand.

In the case of a driven mean field system [25]–[27], the external force is also present in equation (10) and after a transient the system becomes stationary *for any temperature*, i.e. $\mu(t) \equiv \mu$, $\Sigma(t, t') = \Sigma(t - t')$ and $D(t, t') = D(t - t')$. The functions D and Σ depend on the strength α of the driving force. Below $T_{\rm d}$ and for small α they are again related by a generalization of the fluctuation-dissipation theorem similar to the one obtained for $\alpha = 0$.

In this paper we focus on Langevin equations similar to equation (10). Their main characteristic is that the thermal bath, represented by the functions $\Sigma(t, t')$ and $D(t, t')$, is not in equilibrium. As we are interested in the driven case, we restrict to considering stationary functions $\Sigma(t - t')$ and $D(t - t')$.

Let us remark that equation (10) is expected to describe the dynamics of a single Brownian particle immersed in a non-equilibrium environment, or the dynamics of an effective degree of freedom representing a many-particle mean field glassy system. If one wishes to describe in full detail the relaxation of real glasses in finite dimensions, more complicated effects have to be taken into account. For example, the dynamics is expected to be heterogeneous, yielding a *local* effective temperature which may depend on space inside the sample; see e.g. [37] for a detailed discussion. The extension of the results that we shall present in this paper to glassy systems in finite dimensions might require additional work.

1.5. Summary and structure of the paper

The aim of this work is to discuss the validity of the fluctuation theorem for the Langevin equation (10) in the presence of a non-trivial environment represented by the functions $\Sigma(t)$ and $D(t)$. This equation describes a situation in which the relaxing system does not equilibrate with its environment (even in the absence of driving forces) and a non-trivial effective temperature defined from the modification of the fluctuation-dissipation theorem exists. Our first aim is to identify the entropy production rate and to show that the effective temperature of the environment should replace the ambient temperature of a conventional bath.

We investigate analytically as particular cases the dynamics in a harmonic well and, numerically, a case in which the equations of motion are nonlinear. The former problem is relevant for experiments on confined Brownian particles in complex media [34, 38]. In both cases we show that the entropy production rate that we introduced verifies the fluctuation relation in general.

The paper is organized as follows. In section 2 we identify the correct definition of entropy production rate using a rather general procedure proposed by Lebowitz and Spohn. As an illustration, in section 3 we analytically compute the large deviation function in the harmonic case for a white equilibrium bath, already discussed in [6], and for a complex bath. We show that the large deviation function is a convex function of p and satisfies the fluctuation relation. In section 4 we compare the analytic results obtained for the linear problem with numerical data. We also numerically investigate a nonlinear Langevin equation, in which the entropy production is not only given by the work of the

external forces, but should contain an 'internal' term as well. Surprisingly, this term turns out to be negligible, a result we attribute to decorrelation between the work of internal and external forces. In section 5 we make contact with glassy problems and discuss some connections with recent numerical simulations. Section 6 is devoted to the analysis of the link between the modified fluctuation theorem and modified Green–Kubo relations. In section 7 we show explicitly that the distribution of the work done by a slow periodic drive satisfies the fluctuation relation with the low frequency effective temperature, and argue that this procedure is the one that could be 'easily' implemented experimentally as a means to measure the (low frequency) effective temperature. In the conclusions we briefly discuss some experiments that could test our predictions, and compare them to previous studies of the same problem.

2. Entropy production

In this section we introduce the set of dynamic models that we discuss in detail, namely, stochastic processes of generic Langevin type with additive noise. Next, we define the large deviation function that we shall use to test the validity of the fluctuation relation and, finally, we derive a general expression for the entropy production rate.

2.1. The model

In this paper we focus on different aspects of the random motion of a particle in a confining potential, in contact with a thermal environment, and under the effect of a driving external force. The Langevin equation describing the motion of such a particle in a d-dimensional space reads

$$m\ddot{r}_\alpha(t) + \int_{-\infty}^{\infty} \mathrm{d}t' \, g_{\alpha\beta}(t - t')\dot{r}_\beta(t') = -\frac{\partial V}{\partial r_\alpha}[\vec{r}(t)] + \rho_\alpha(t) + h_\alpha(t), \qquad \alpha = 1, \ldots, d. \quad (11)$$

$\vec{r} = (r_1, \ldots, r_\mathrm{d})$ is the position of the particle. We pay special attention to the case $d = 2$ and call (x, y) the two components of the position vector. m is the mass of the particle and $V(\vec{r})$ is a potential energy. As an example we shall work out in detail the simple harmonic case, $V(\vec{r}) = (k/2) \sum_\alpha r_\alpha^2$ with k the spring constant of the quadratic potential. $\vec{\rho}(t)$ is a Gaussian thermal noise with zero average and generic stationary correlation

$$\langle \, \rho_\alpha(t)\rho_\beta(t') \, \rangle = \delta_{\alpha\beta} \, \nu(t - t') \qquad \alpha, \beta = 1, \ldots, d, \quad (12)$$

with $\nu(t - t')$ a symmetric function of $t - t'$. The memory kernel $g_{\alpha\beta}(t - t')$ extends the notion of friction to a more generic case. We assume a simple spatial structure, $g_{\alpha\beta}(t - t') = \delta_{\alpha\beta} \, g(t - t')$. In order to ensure causality we take $g(t - t')$ to be proportional to $\theta(t - t')$. We define

$$g(t) = \theta(t)f(t) \qquad \text{with } f(t) \equiv g(t) + g(-t). \quad (13)$$

The initial time t_0 has been taken to $-\infty$. $\vec{h}(t)$ is a time dependent field that either we use to compute the linear response or represents the external forcing.

It will be useful to introduce Fourier transforms. We use the conventions

$$\rho(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, \mathrm{e}^{-\mathrm{i}\omega t} \, \rho(\omega), \qquad \rho(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{e}^{\mathrm{i}\omega t} \, \rho(t). \quad (14)$$

The fluctuation-dissipation theorem states that for systems evolving in thermal equilibrium with their equilibrated environment the linear response is related to the correlation function of the same observable as

$$R(t) = -\frac{\theta(t)}{T}\frac{\mathrm{d}C(t)}{\mathrm{d}t}, \qquad \frac{\mathrm{d}C(t)}{\mathrm{d}t} = T[R(-t) - R(t)]; \tag{15}$$

see equations (4) and (5) for the definitions of R and C. Here and in what follows we set the Boltzmann constant to one, $k_\mathrm{B} = 1$. To write the second expression we used that $C(t)$ is an even function of t (and $\dot{C}(t)$ is odd) and defined $\theta(0) \equiv 1/2$ (the same convention is used in the rest of the paper). After Fourier transforming, the second expression becomes

$$\omega C(\omega) = 2T \operatorname{Im} R(\omega), \tag{16}$$

and the real part of $R(\omega)$ is related to $\operatorname{Im} R(\omega)$ by the Kramers–Krönig relation.

The functions g and ν are the integrated response and correlation of the bath, respectively. As will be discussed in detail in section 3, if the bath is itself in equilibrium at a temperature T, they are related as

$$\frac{1}{T} = \frac{2 \operatorname{Re} g(\omega)}{\nu(\omega)}. \tag{17}$$

For an out-of-equilibrium bath we define the frequency dependent temperature

$$\frac{1}{T(\omega)} = \frac{2 \operatorname{Re} g(\omega)}{\nu(\omega)}, \tag{18}$$

and its inverse Fourier transform

$$T^{-1}(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{2 \operatorname{Re} g(\omega)}{\nu(\omega)} \mathrm{e}^{-\mathrm{i}\omega t} = \int_{-\infty}^{\infty} \mathrm{d}t'\, \nu^{-1}(t - t') f(t'), \tag{19}$$

where

$$\nu^{-1}(t) = \int \frac{\mathrm{d}\omega}{2\pi} \frac{1}{\nu(\omega)} \mathrm{e}^{-\mathrm{i}\omega t}. \tag{20}$$

Note that both $f(t)$ and $T^{-1}(t)$ are even functions of t. If the bath is in equilibrium at temperature T, $T^{-1}(t - t') = \delta(t - t')/T$. We shall assume throughout that $T^{-1}(t)$ goes to zero fast enough for large t.

The main result of this paper is that the fluctuation relation for the probability distribution function $\pi_\tau(p)$ holds for long times τ:

$$\tau\sigma_+\, p \sim \ln \frac{\pi_\tau(p)}{\pi_\tau(-p)}, \tag{21}$$

with

$$\mathcal{S}_\tau = \tau\sigma_+ p = \int_{-\tau/2}^{\tau/2} \mathrm{d}t \int_{-\tau/2}^{\tau/2} \mathrm{d}t'\, T^{-1}(t - t')\, \dot{r}_\alpha(t) \left[-\frac{\partial V}{\partial r_\alpha}[\vec{r}(t')] + h_\alpha[\vec{r}(t')] \right]. \tag{22}$$

2.2. Large deviation function

In the rest of this section we identify the entropy production rate for equation (11) following the procedure proposed by Lebowitz and Spohn [7].

The fluctuation relation is a symmetry property of the probability distribution function (PDF) of the entropy production rate σ_t that can also be expressed as a symmetry of the large deviation function. Writing σ_+ for the average value of σ_t, the PDF of the variable p defined in equation (1) is defined as

$$\pi_\tau(p) = \mathcal{P}\left[\frac{\mathcal{S}_\tau}{\tau\sigma_+} = p\right] = \tau\sigma_+\langle\delta(\mathcal{S}_\tau - p\tau\sigma_+)\rangle, \tag{23}$$

where the angular brackets denote an average over the realizations of the noise. The large deviation function (normalized so that $\zeta(1) = 0$ at the maximum) is given by

$$\zeta(p) = \lim_{\tau\to\infty} \tau^{-1} \ln\left[\pi_\tau(p)/\pi_\tau(1)\right]. \tag{24}$$

It is easier to compute the characteristic function

$$\phi(\lambda) = -\lim_{\tau\to\infty} \tau^{-1} \ln\langle\exp[-\lambda\sigma_\tau]\rangle, \tag{25}$$

the latter being the Legendre transform of $\zeta(p)$. Indeed,

$$e^{-\tau\phi(\lambda)} = \langle e^{-\lambda\sigma_\tau}\rangle = \frac{\int dp\, e^{\tau[\zeta(p)-\lambda p\sigma_+]}}{\int dp\, e^{\tau\zeta(p)}} \sim \frac{e^{\tau\max_p[\zeta(p)-\lambda p\sigma_+]}}{e^{\tau\zeta(1)}} \tag{26}$$

so that, recalling that $\zeta(1) = 0$ by construction,

$$\phi(\lambda) = -\max_p\left[\zeta(p) - \lambda p\sigma_+\right]. \tag{27}$$

The inversion of the Legendre transform yields

$$\zeta(p) = \min_\lambda\left[\lambda p\sigma_+ - \phi(\lambda)\right], \tag{28}$$

and it is easy to check that the fluctuation relation is equivalent to $\phi(\lambda) = \phi(1-\lambda)$.

2.3. Internal symmetries and the fluctuation relation

Assume that there exists a map I on the space of trajectories $r(t)$ such that $I^2 = 1$ and that the measure $\mathcal{D}r$ is invariant under I, i.e. $\mathcal{D}Ir = \mathcal{D}r$. Then, consider a segment of trajectory $r(t)$, $t \in [-\tau/2, \tau/2]$, and define

$$\mathcal{S}_\tau = -\ln\frac{\mathcal{P}[Ir(t)]}{\mathcal{P}[r(t)]}, \tag{29}$$

where $\mathcal{P}[r(t)]$ is the probability of observing $r(t)$, *in the stationary state*, for $t \in [-\tau/2, \tau/2]$ irrespectively of what happens outside the interval $[-\tau/2, \tau/2]$. It is easy to show that the PDF of $\mathcal{S}_\tau$ verifies the fluctuation theorem. Indeed

$$\langle e^{-\lambda\mathcal{S}_\tau}\rangle = \int \mathcal{D}r\, \mathcal{P}[r(t)]e^{-\lambda\mathcal{S}_\tau} = \int \mathcal{D}r\, \mathcal{P}[r(t)]^{1-\lambda}\, \mathcal{P}[Ir(t)]^\lambda$$

$$= \int \mathcal{D}r\, \mathcal{P}[Ir(t)]^{1-\lambda}\, \mathcal{P}[r(t)]^\lambda = \langle e^{-(1-\lambda)\mathcal{S}_\tau}\rangle. \tag{30}$$

Thus, if the limit

$$\phi(\lambda) = -\lim_{\tau \to \infty} \tau^{-1} \ln \langle e^{-\lambda \mathcal{S}_\tau} \rangle \tag{31}$$

exists, it verifies the relation $\phi(\lambda) = \phi(1 - \lambda)$ from which the fluctuation relation for the PDF of $\mathcal{S}_\tau$ follows. Lebowitz and Spohn showed that the limit $\phi(\lambda)$ indeed exists for generic Markov processes and it is a concave function of λ. Moreover, they showed that $\mathcal{S}_\tau$ can be identified with the entropy production rate—over the time interval τ—in the stationary state up to boundary terms, i.e. terms that do not grow with τ, if I is chosen to be the time reversal, $Ir(t) = r(-t)$.

2.4. Entropy production rate

We are interested in the explicit form of $\mathcal{S}_\tau$ for the equation of motion (11) in the case in which $g_{\alpha\beta}(t) = \delta_{\alpha\beta} g(t)$ and $\vec{h}(t) = \vec{h}[\vec{r}(t)]$ is an external non-conservative force that does not explicitly depend on time: e.g., for $d = 2$, $\vec{h} = \alpha(-y, x)$. Note that the functions $\nu(t)$ and $g(t)$ are such that $\nu(t) = \nu(-t)$ while $g(t)$ is proportional to $\theta(t)$, and both decay sufficiently rapidly in time. The probability distribution of the noise $\vec{\rho}(t)$ is

$$\mathcal{P}[\vec{\rho}(t)] \propto \exp\left[-\tfrac{1}{2} \int dt\, dt'\, \rho_\alpha(t) \nu^{-1}(t - t') \rho_\alpha(t') \right], \tag{32}$$

where $\nu^{-1}(t)$ is the operator inverse of $\nu(t)$; see equation (20). The probability distribution of $\vec{r}(t)$ is obtained by substituting $\vec{\rho}(t)$ obtained from equation (11) in equation (32). One has

$$\mathcal{P}[\vec{r}(t)] \propto \exp\left\{ -\frac{1}{2} \int dt\, dt' \left[m\ddot{r}_\alpha(t) + \int dt''\, g(t - t')\dot{r}_\alpha(t'') + \frac{\partial V}{\partial r_\alpha}[\vec{r}(t)] - h_\alpha[\vec{r}(t)] \right] \right.$$
$$\left. \times \nu^{-1}(t - t') \left[m\ddot{r}_\alpha(t') + \int dt'''\, g(t' - t''')\dot{r}_\alpha(t''') + \frac{\partial V}{\partial r_\alpha}[\vec{r}(t')] - h_\alpha[\vec{r}(t')] \right] \right\}. \tag{33}$$

After some algebra it is easy to see that

$$\mathcal{P}[\vec{r}(-t)] \propto \exp\left\{ -\frac{1}{2} \int dt\, dt' \left[m\ddot{r}_\alpha(t) - \int dt''\, g(t'' - t)\dot{r}_\alpha(t'') + \frac{\partial V}{\partial r_\alpha}[\vec{r}(t)] - h_\alpha[\vec{r}(t)] \right] \right.$$
$$\left. \times \nu^{-1}(t - t') \left[m\ddot{r}_\alpha(t') - \int dt'''\, g(t''' - t')\dot{r}_\alpha(t''') + \frac{\partial V}{\partial r_\alpha}[\vec{r}(t')] - h_\alpha[\vec{r}(t')] \right] \right\}. \tag{34}$$

To compute $\mathcal{S}_\tau$ we should consider the probability of a segment of trajectory $[-\tau/2, \tau/2]$ and then send τ to ∞, neglecting all boundary terms. As the functions $g(t)$ and $\nu(t)$ have short range, the trajectories $\vec{r}(t)$ decorrelate, say, exponentially fast in time and up to boundary contributions one can simply truncate the integrals in $\mathcal{P}[\vec{r}(t)]$ in $t, t' \in [-\tau/2, \tau/2]$.

Let us now discuss the contributions to $-\ln \mathcal{P}[\vec{r}(-t)] + \ln \mathcal{P}[\vec{r}(t)]$ that do not trivially vanish. One has:

- A 'kinetic' term of the form

$$
\int \mathrm{d}t\, \mathrm{d}t' \left[m\ddot{r}_\alpha(t)\nu^{-1}(t-t') \int \mathrm{d}t''\, g(t''-t')\dot{r}_\alpha(t'') \right.
$$

$$
\left. + m\ddot{r}_\alpha(t)\nu^{-1}(t-t') \int \mathrm{d}t''\, g(t'-t'')\dot{r}_\alpha(t'') \right]
$$

$$
= \int \mathrm{d}t\, \mathrm{d}t'\, m\ddot{r}_\alpha(t)\nu^{-1}(t-t') \int \mathrm{d}t''\, f(t'-t'')\dot{r}_\alpha(t'')
$$

$$
= \int \mathrm{d}t\, \mathrm{d}t'\, m\ddot{r}_\alpha(t)T^{-1}(t-t')\dot{r}_\alpha(t'). \tag{35}
$$

If the bath is in equilibrium, this term trivially vanishes as it is the integral of the total derivative of the kinetic energy. But it also vanishes for a non-equilibrium bath. Indeed, by integrating by parts first in t and then in t', we find

$$
\int \mathrm{d}t\, \mathrm{d}t'\, \ddot{r}_\alpha(t)T^{-1}(t-t')\dot{r}_\alpha(t') = -\int \mathrm{d}t\, \mathrm{d}t'\, \dot{r}_\alpha(t)\frac{\mathrm{d}}{\mathrm{d}t}T^{-1}(t-t')\dot{r}_\alpha(t')
$$

$$
= \int \mathrm{d}t\, \mathrm{d}t'\, \dot{r}_\alpha(t)\frac{\mathrm{d}}{\mathrm{d}t'}T^{-1}(t-t')\dot{r}_\alpha(t')
$$

$$
= -\int \mathrm{d}t\, \mathrm{d}t'\, \dot{r}_\alpha(t)T^{-1}(t-t')\ddot{r}_\alpha(t') = 0, \tag{36}
$$

where we used that $T^{-1}(t)$ is even and short ranged and we neglected boundary terms.

- A 'friction' term of the form

$$
\tfrac{1}{2}\int \mathrm{d}t\, \mathrm{d}t'\, \mathrm{d}t''\, \mathrm{d}t''' \left[\dot{r}_\alpha(t'')g(t''-t)\nu^{-1}(t-t')g(t'''-t')\dot{r}_\alpha(t''') \right.
$$

$$
\left. - \dot{r}_\alpha(t'')g(t-t'')\nu^{-1}(t-t')g(t'-t''')\dot{r}_\alpha(t''') \right]. \tag{37}
$$

This term vanishes because the function

$$
K(t''-t''') = \int \mathrm{d}t\, \mathrm{d}t'\, g(t''-t)\nu^{-1}(t-t')g(t'''-t') \tag{38}
$$

is even in its argument as one can easily check.

- A 'potential' term of the form

$$
\mathcal{S}_\tau^V = -\int_{-\tau/2}^{\tau/2} \mathrm{d}t \int_{-\tau/2}^{\tau/2} \mathrm{d}t' \int_{-\tau/2}^{\tau/2} \mathrm{d}t''\, f(t-t'')\dot{r}_\alpha(t'')\nu^{-1}(t-t')\frac{\partial V}{\partial r_\alpha}[\vec{r}(t')]
$$

$$
= -\int_{-\tau/2}^{\tau/2} \mathrm{d}t \int_{-\tau/2}^{\tau/2} \mathrm{d}t'\, T^{-1}(t-t')\dot{r}_\alpha(t)\frac{\partial V}{\partial r_\alpha}[\vec{r}(t')]. \tag{39}
$$

This term is related to the work of the conservative forces. If the bath is in equilibrium, it vanishes, being the total derivative of the potential energy. It vanishes also for a harmonic potential $V(\vec{r}) = \frac{1}{2}kr^2$ because $(\partial V/\partial r_\alpha)[\vec{r}(t)] = kr_\alpha(t)$ and one can use the same trick as was used in equation (36). It does not vanish in general.

- A 'dissipative' term

$$
\mathcal{S}_\tau^{\mathrm{diss}} = \int_{-\tau/2}^{\tau/2} \mathrm{d}t \int_{-\tau/2}^{\tau/2} \mathrm{d}t'\, T^{-1}(t-t')\dot{r}_\alpha(t)h_\alpha[\vec{r}(t')]. \tag{40}
$$

This term is related to the work of the dissipative forces. If the bath is in equilibrium at temperature T, this is exactly the work of the dissipative forces divided by the temperature of the bath. Otherwise, the work done at frequency ω is weighted by the effective temperature at the same frequency.

The expression for the total entropy production over the interval $[-\tau/2, \tau/2]$ is then

$$
\begin{aligned}
\mathcal{S}_\tau = \mathcal{S}_\tau^V + \mathcal{S}_\tau^{\text{diss}} &= \int_{-\tau/2}^{\tau/2} \mathrm{d}t \int_{-\tau/2}^{\tau/2} \mathrm{d}t' \, T^{-1}(t-t') \dot{r}_\alpha(t) \left[-\frac{\partial V}{\partial r_\alpha}[\vec{r}(t')] + h_\alpha[\vec{r}(t')] \right] \\
&= \int_{-\tau/2}^{\tau/2} \mathrm{d}t \int_{-\tau/2}^{\tau/2} \mathrm{d}t' \, T^{-1}(t-t') \dot{r}_\alpha(t) F_\alpha(t'),
\end{aligned}
\tag{41}
$$

where $F_\alpha(t) = h_\alpha[\vec{r}(t)] - (\partial V/\partial r_\alpha)[\vec{r}(t)]$ is the total deterministic force acting on the particle at time t.

The latter expression can be rewritten as

$$
\mathcal{S}_\tau \sim \int_{-\tau/2}^{\tau/2} \mathrm{d}t \, \sigma_t = \int_{-\tau/2}^{\tau/2} \mathrm{d}t \, (\sigma_t^V + \sigma_t^{\text{diss}})
\tag{42}
$$

defining entropy production rates σ_t, σ_t^V and σ_t^{diss} (modulo subdominant terms in the large τ limit) as

$$
\begin{aligned}
\sigma_t = \sigma_t^V + \sigma_t^{\text{diss}} &= \int_{-\infty}^{t} \mathrm{d}t' \, T^{-1}(t-t')[\dot{r}_\alpha(t) F_\alpha(t') + \dot{r}_\alpha(t') F_\alpha(t)], \\
\sigma_t^V &= -\int_{-\infty}^{t} \mathrm{d}t' \, T^{-1}(t-t') \left[\dot{r}_\alpha(t) \frac{\partial V}{\partial r_\alpha}[\vec{r}(t')] + \dot{r}_\alpha(t') \frac{\partial V}{\partial r_\alpha}[\vec{r}(t)] \right], \\
\sigma_t^{\text{diss}} &= \int_{-\infty}^{t} \mathrm{d}t' \, T^{-1}(t-t')[\dot{r}_\alpha(t) h_\alpha[\vec{r}(t')] + \dot{r}_\alpha(t') h_\alpha[\vec{r}(t)]].
\end{aligned}
\tag{43}
$$

We recall that if the bath is in equilibrium this expression reduces to the work done by the non-conservative forces divided by the temperature of the bath, as expected. If the bath is not in equilibrium, but the potential is harmonic, only the contribution $\sigma_\tau^{\text{diss}}$ of the non-conservative force has to be taken into account. The reason that the work of the conservative forces produces entropy if the bath is out of equilibrium *and* the interaction is nonlinear is that the nonlinear interaction couples modes of different frequency which are at different temperature, thus producing an energy flow between these modes; this energy flow is related to the entropy production.

It is also important to remark that boundary terms, which are usually neglected, can have dramatic effects on the large fluctuations of $\mathcal{S}_\tau$ even for $\tau \to \infty$, as pointed out by Van Zon and Cohen [40]. This happens if the PDF of the boundary term has exponential tails. Thus, boundary terms *cannot always be neglected*, at least for very large values of p. A good empirical prescription for removing boundary contributions is the following: in equilibrium σ_t must be a total derivative as no dissipation is present; so, removing a total derivative (a boundary term), one can define $\mathcal{S}_\tau$ in such a way that it vanishes identically in equilibrium. This definition turns out to be the one that verifies the fluctuation relation for all $|p| < p^*$, p^* being the maximum allowed value of p for $\tau \to \infty$ [40, 41]. We shall discuss this point in more detail later.

14

3. A driven particle in a harmonic potential

In this and the next section we discuss some examples on which we test the fluctuation relation for $\mathcal{S}_\tau$ defined in (41). We first consider the simplest fully analytically solvable case in which there are no applied forces and the potential is quadratic. We derive the fluctuation-dissipation relation between induced and spontaneous fluctuations in the position of the particle and we relate it to the time dependent temperature of the bath defined in equation (19). We then show that in this simple case the fluctuation relation for $\mathcal{S}_\tau$ in (41) reduces to the usual one with the temperature of the bath.

3.1. The fluctuation-dissipation relation

In the harmonic Brownian particle problem with no other applied external forces the dynamics of different spatial components are not coupled. Thus, without loss of generality, we henceforth focus on $d = 1$. In Fourier space, the Langevin equation reads

$$-m\omega^2 x(\omega) - \mathrm{i}\omega g(\omega)x(\omega) = -kx(\omega) + \rho(\omega) \tag{44}$$

with the noise–noise correlation

$$\langle\, \rho(\omega)\rho(\omega') \,\rangle = 2\pi\delta(\omega + \omega')\nu(\omega). \tag{45}$$

The linear equation (44) is solved by

$$x(\omega) = G(\omega)\rho(\omega), \qquad G(\omega) \equiv \frac{1}{-m\omega^2 - \mathrm{i}\omega g(\omega) + k}, \tag{46}$$

and one finds the correlations

$$\begin{aligned}
\langle\, x(\omega)x(\omega') \,\rangle &= G(\omega)G(-\omega)2\pi\delta(\omega + \omega')\nu(\omega), \\
\langle\, x(\omega)\rho(\omega') \,\rangle &= G(\omega)2\pi\delta(\omega + \omega')\nu(\omega).
\end{aligned} \tag{47}$$

Note that $G(\omega)G(-\omega) = |G(\omega)|^2$; then

$$\langle\, x(\omega)x(\omega') \,\rangle = C(\omega)2\pi\delta(\omega + \omega') \qquad \text{with } C(\omega) \equiv |G(\omega)|^2\nu(\omega). \tag{48}$$

In a problem solved by

$$x(t) = \int_{-\infty}^{\infty} \mathrm{d}t'\, G(t - t')[\rho(t') + h(t')] + \mathrm{IC}, \tag{49}$$

where IC are terms related to the initial conditions, the time dependent linear response is

$$R(t - t') \equiv \frac{\delta\langle x(t)\rangle}{\delta h(t')}\bigg|_{h=0} = G(t - t'), \tag{50}$$

and

$$R(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}R(t) = G(\omega). \tag{51}$$

Note that the response function is related to the correlation $\langle\, x(t)\rho(t') \,\rangle$ by equation (47):

$$2\pi\delta(\omega + \omega')R(\omega)\nu(\omega) = \langle\, x(\omega)\rho(\omega') \,\rangle. \tag{52}$$

15

Now, we can check under which conditions on the characteristics of the bath ($g(t-t')$ and $\nu(t-t')$) the fluctuation-dissipation theorem (for the particle) holds and, when it does not hold, which is the generic form that the relation between the linear response and correlation might take in this simple quadratic model. Equation (46) implies[9]

$$\text{Im}\, R(\omega) = \text{Im}\, G(\omega) = \omega\ \text{Re}\, g(\omega)\ |G(\omega)|^2, \tag{53}$$

and then using equation (48)

$$\frac{\omega C(\omega)}{2\,\text{Im}\, R(\omega)} = \frac{\nu(\omega)}{2\,\text{Re}\, g(\omega)} = T(\omega). \tag{54}$$

We see that the fluctuation-dissipation theorem holds only if this ratio is equal to T; see equation (16). Otherwise, the relation between the linear response and correlation of the particle is given by the frequency dependent temperature of the bath, $T(\omega)$, defined in (18). The measure of the modification of the fluctuation-dissipation theorem given in (54) is the *effective temperature* of the system. The use of this name has been justified within a number of models with slow dynamics and a separation of timescales [9, 10] but it might not hold in complete generality [11]. It is important to remark that the effective temperature in the frequency domain, $T(\omega)$, is *not* equal in general to the Fourier transform of the effective temperature $T_{\text{eff}}(t)$—defined in the introduction—which is the ratio between the noise–noise correlation and memory function in the time domain.

Let us now discuss some environments that we shall use in the rest of the paper.

3.1.1. Equilibrated environments. For any environment such that the right-hand side in equation (54) equals T the fluctuation-dissipation theorem holds. In the time domain, this condition reads

$$T g(t) = \theta(t)\nu(t), \qquad \nu(t) = T[g(t) + g(-t)] = T g(|t|). \tag{55}$$

In particular, this is satisfied by a white noise for which $\nu(t) = 2T\gamma\delta(t)$ and $g(t) = 2\gamma\delta(t)\theta(t)$ (remember that $\theta(0) \equiv 1/2$). The fluctuation-dissipation theorem also holds for any coloured noise—with a retarded memory kernel g and noise–noise correlation ν—such that the ratio between $\text{Re}\, g(\omega)$ and $\nu(\omega)$ equals $(2T)^{-1}$. This requirement applies to any equilibrated bath.

3.1.2. Non-equilibrium environments. Instead, for any other generic environment, the left-hand side in equation (54) yields a non-trivial and, in general, model dependent result for the effective temperature.

A special case that we shall study in appendix B is the one of an ensemble of N equilibrated baths with different characteristic times and at different temperatures. In this case, the noise $\vec{\rho}$ in equation (11) is the sum of N independent noises,

$$\vec{\rho} = \sum_{i=1}^{N} \vec{\rho}_i, \qquad \langle\, \rho_{i\alpha}(t)\rho_{j\beta}(t') \,\rangle = \delta_{\alpha\beta}\delta_{ij}T_i\nu_i(t - t'), \tag{56}$$

---

[9] The discussion in these paragraphs has to be modified in the $k = 0$ limit in which the particle does not have a confining potential and diffuses.

and the friction kernel is given by

$$g(t - t') = \sum_{i=1}^{N} g_i(t - t'). \tag{57}$$

We have extracted the temperature T_i from the definition of $\nu_i(t)$ in order to simplify several expressions. As the $\vec{\rho}_i$ are independent Gaussian variables, $\vec{\rho} = \sum_i \vec{\rho}_i$ is still a Gaussian variable with zero mean and correlation

$$\langle \rho_\alpha(t) \rho_\beta(t') \rangle = \delta_{\alpha\beta} \sum_i T_i \nu_i(t - t'). \tag{58}$$

Thus, in the Gaussian case *the N equilibrated baths are equivalent to a single non-equilibrium bath* with correlation given by equation (58) and friction kernel given by equation (57). In frequency space we have

$$g(\omega) = \sum_{i=1}^{N} g_i(\omega), \qquad \nu(\omega) = \sum_{i=1}^{N} T_i \nu_i(\omega), \tag{59}$$

with

$$\nu_i(\omega) = 2 \operatorname{Re} g_i(\omega), \tag{60}$$

as each bath is equilibrated at temperature T_i. The frequency dependent temperature is then given by

$$T(\omega) = \frac{\sum_{i=1}^{N} T_i \nu_i(\omega)}{\sum_{i=1}^{N} \nu_i(\omega)}. \tag{61}$$

Note that if the functions $\nu_i(\omega)$ are chosen to be peaked around a frequency ω_i, by choosing suitable values of ω_i and T_i one can approximate a single non-equilibrium bath with N baths equilibrated at different temperatures.

3.2. Large deviation function

We now compute the large deviation function in the harmonic case, $V(\vec{r}) = \frac{1}{2}kr^2$. In this case σ_t^V is a total derivative and only the term σ_t^{diss}, related to the non-conservative forces, is relevant. We show that the characteristic function $\phi(\lambda)$ of σ_t^{diss} exists, is a convex function of λ and verifies the fluctuation relation $\phi(\lambda) = \phi(1 - \lambda)$.

3.2.1. Equilibrium bath. As a first illustrative example we consider the case of an equilibrium white bath. The model we study is a two-dimensional harmonic oscillator with potential energy $V(x, y) = (k/2)(x^2 + y^2)$ coupled to a simple white bath in equilibrium at temperature T, and driven out of equilibrium by the non-conservative force $\vec{h} = \alpha(-y, x)$. The equations of motion are

$$\begin{aligned} m\ddot{x}_t + \gamma\dot{x}_t &= -kx_t - \alpha y_t + \xi_t, \\ m\ddot{y}_t + \gamma\dot{y}_t &= -ky_t + \alpha x_t + \eta_t, \end{aligned} \tag{62}$$

where ξ_t and η_t are independent Gaussian white noises with variance $\langle \xi_t \xi_0 \rangle = \langle \eta_t \eta_0 \rangle = 2\gamma T \delta(t)$. The memory friction kernels $g_{\alpha\beta}(t-s)$ are simply $\delta_{\alpha\beta} g(t-s) = 2\delta_{\alpha\beta} \gamma \delta(t-s)\theta(t-s)$ in this case, with γ the friction coefficient.

Defining the complex variable $z_t = (x_t + \mathrm{i} y_t)/\sqrt{2}$ and the noise $\rho_t = (\xi_t + \mathrm{i}\eta_t)/\sqrt{2}$ the equations of motion can be written as

$$m\ddot{z}_t + \gamma \dot{z}_t = -\kappa z_t + \rho_t, \tag{63}$$

where $\kappa = k - \mathrm{i}\alpha$, $\langle \rho_t \rho_0 \rangle = \langle \bar{\rho}_t \bar{\rho}_0 \rangle = 0$ and $\langle \rho_t \bar{\rho}_0 \rangle = 2\gamma T \delta(t)$. The complex noise ρ_t has a Gaussian PDF:

$$\mathcal{P}[\rho_t] \propto \exp\left[-\frac{1}{2\gamma T} \int_{-\infty}^{\infty} \mathrm{d}t \; \rho_t \bar{\rho}_t \right]. \tag{64}$$

The energy of the oscillator is $H = m\dot{z}\dot{\bar{z}} + kz\bar{z}$, and its time derivative is given by

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 2m\,\mathrm{Re}\,\dot{z}_t \ddot{\bar{z}}_t + 2k\,\mathrm{Re}\,z_t \dot{\bar{z}}_t = 2\alpha \mathrm{Im} \dot{z}_t \bar{z}_t - 2\gamma \dot{z}_t \dot{\bar{z}}_t + 2\,\mathrm{Re}\,\dot{z}_t \bar{\rho}_t = W_t - \widetilde{W}_t, \tag{65}$$

where $W_t = 2\alpha\,\mathrm{Im}\,\dot{z}_t \bar{z}_t = \alpha(x_t \dot{y}_t - y_t \dot{x}_t)$ is the power injected by the driving force and $\widetilde{W}_t = 2\gamma \dot{z}_t \dot{\bar{z}}_t - 2\,\mathrm{Re}\,\dot{z}_t \bar{\rho}_t$ is the power extracted by the thermostat (henceforth we choose the sign of the power in order to have positive average).

The entropy production rate (43) reduces, as expected, to the injected power divided by the temperature, $\sigma_t = \beta W_t$, where $\beta = 1/T$ (one could also consider the entropy production of the bath, $\widetilde{\sigma}_t = \beta \widetilde{W}_t$; for completeness we discuss it in appendix C).

We want to compute the probability distribution function (PDF) of the entropy production rate $\sigma_t = \beta W_t$. The average value of σ_t is in this case given by $\sigma_+ = 2\alpha^2/(\gamma k)$. From equation (41) we can rewrite the total entropy production $\mathcal{S}_\tau$ in terms of the complex variable z_t:

$$\mathcal{S}_\tau = \int_{-\tau/2}^{\tau/2} \mathrm{d}t\; \sigma_t = 2\alpha\beta\;\mathrm{Im} \int_{-\tau/2}^{\tau/2} \mathrm{d}t\; \dot{z}_t \bar{z}_t. \tag{66}$$

As already discussed, it is easier to compute the characteristic function $\phi(\lambda)$, equation (31), in terms of which the fluctuation relation reads $\phi(\lambda) = \phi(1-\lambda)$. To leading order in τ we can neglect all the boundary terms in the integrals. After integrating by parts we have

$$\mathcal{S}_\tau = 2\alpha\beta\mathrm{i} \int_{-\tau/2}^{\tau/2} \mathrm{d}t\; z_t \dot{\bar{z}}_t. \tag{67}$$

In terms of the PDF of the noise (64) we obtain

$$\langle \exp[-\lambda \mathcal{S}_\tau] \rangle = \mathcal{N}^{-1} \int \mathrm{d}\mathcal{P}[\rho_t]\; \exp\left[-\frac{2\mathrm{i}\alpha\lambda}{T} \int_{-\tau/2}^{\tau/2} \mathrm{d}t\; z_t \dot{\bar{z}}_t \right], \tag{68}$$

and the normalization factor $\mathcal{N} = \int \mathrm{d}\,\mathcal{P}[\rho_t]$ is simply given by the numerator calculated at $\lambda = 0$.

To leading order in τ the function $\phi(\lambda)$ should not depend on the boundary conditions in equation (68). Thus, we impose periodic boundary conditions, $z(\tau/2) = z(-\tau/2)$ and

$\dot{z}(\tau/2) = \dot{z}(-\tau/2)$, and we expand z_t in a Fourier series,

$$z_t = \frac{\Delta\omega}{2\pi} \sum_{n=-\infty}^{\infty} z_n \, \mathrm{e}^{-\mathrm{i}\omega_n t}, \tag{69}$$

where $\Delta\omega = 2\pi/\tau$ and $\omega_n = n\Delta\omega$. For $\tau \to \infty$,

$$z_t = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, \mathrm{e}^{-\mathrm{i}\omega t} z_\omega, \qquad z_\omega = \int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{e}^{\mathrm{i}\omega t} z_t, \tag{70}$$

and the equations of motion become

$$z_\omega = \frac{\rho_\omega}{-\omega^2 m + \kappa - \mathrm{i}\omega\gamma} \equiv \frac{\rho_\omega}{D(\omega)}. \tag{71}$$

Note that in the limit $\alpha = 0$ the Green function $G(\alpha, \omega) = 1/D(\omega)$ reduces to the one used above to compute the violation of the fluctuation-dissipation theorem induced by a non-equilibrium bath. The distribution of the noise is given by

$$\mathcal{P}[\rho_\omega] = \exp\left[-\frac{1}{2\gamma T} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, \rho_\omega \bar{\rho}_\omega\right] \sim \exp\left[-\frac{1}{2\gamma T} \frac{\Delta\omega}{2\pi} \sum_{n=-\infty}^{\infty} \rho_n \bar{\rho}_n\right]. \tag{72}$$

Substituting equations (69) and (72) into (68) we get

$$\langle \exp[-\lambda \mathcal{S}_\tau] \rangle = \mathcal{N}^{-1} \int \mathrm{d}\rho_n \exp\left[-\frac{\Delta\omega}{2\pi} \sum_{n=-\infty}^{\infty} \left(\frac{|\rho_n|^2}{2\gamma T} - \frac{2\alpha\lambda\omega_n |\rho_n|^2}{T|D(\omega_n)|^2}\right)\right]$$

$$= \prod_{n=-\infty}^{\infty} \left[1 - \frac{4\gamma\alpha\lambda\omega_n}{|D(\omega_n)|^2}\right]^{-1} \tag{73}$$

and using equation (25),

$$\phi(\lambda) = \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{n=-\infty}^{\infty} \ln\left[1 - \frac{4\gamma\alpha\lambda\omega_n}{|D(\omega_n)|^2}\right] = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \ln\left[1 - \frac{4\gamma\alpha\lambda\omega}{|D(\omega)|^2}\right]. \tag{74}$$

To show that ϕ verifies $\phi(\lambda) = \phi(1-\lambda)$ and hence the fluctuation theorem, note that

$$\phi(\lambda) - \phi(1-\lambda) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \ln\left[\frac{|D(\omega)|^2 - 4\gamma\alpha\lambda\omega}{|D(\omega)|^2 - 4\gamma\alpha(1-\lambda)\omega}\right] \tag{75}$$

and, as $|D(\omega)|^2 - 4\alpha\gamma\omega = |D(-\omega)|^2$, the integrand is an odd function of ω and the integral vanishes by symmetry. In appendix A we show that the same result is obtained if one uses Dirichlet boundary conditions (at least for $m = 0$, where the computation is feasible); this result supports the approximations made when neglecting all the boundary terms in the exponential. Moreover, in the case $m = 0$ the large deviation function $\zeta(p)$ can be explicitly calculated; defining $\tau_0 = \gamma/k$ and $\sigma_0 = \sigma_+\tau_0/2 = \alpha^2/k^2$, we obtain

$$\zeta(p) = \tau_0^{-1}\left[1 + p\sigma_0 - \sqrt{(1+\sigma_0)(1+p^2\sigma_0)}\right]. \tag{76}$$

Thus, for $\tau \to \infty$ the PDF of p has the form

$$\pi_\tau(p) \propto \exp\left[\frac{\tau}{\tau_0}\tilde{\zeta}(p, \sigma_0)\right]. \tag{77}$$

Note that τ_0 is the decay time of the correlation function of z_t (i.e. $\langle z_t \bar{z}_0 \rangle \propto \exp(-t/\tau_0)$) and σ_0 is the average entropy production over a time $\tau_0/2$. Thus, τ_0 is the natural time unit of the problem (as expected); remarkably, the function $\tilde{\zeta} = \tau_0 \zeta$ depends only on σ_0 and not on the details of the model. It would be interesting to see whether the same scaling holds for more realistic models.

In summary, we found that for all driving forces, i.e. all values of α, the fluctuation theorem holds for the entropy production rate (41). For a white equilibrium bath this result has already been obtained in [6]. The temperature entering the fluctuation theorem is that of the equilibrated environment with which the system is in contact, although it is not in equilibrium with it when the force is applied.

Let us also stress that one can easily check that the fluctuation-dissipation relation holds in the absence of the drive (see section 3.1) but it is strongly violated when the system is taken out of equilibrium by the external force.

3.2.2. Non-equilibrium bath. We now generalize the calculation to the case of a generic non-equilibrium bath; the equation of motion becomes

$$m\ddot{z}_t + \int_{-\infty}^{\infty} \mathrm{d}t' \ g(t-t')\dot{z}_{t'} = -\kappa z_t + \rho_t, \tag{78}$$

where as before $\kappa = k - \mathrm{i}\alpha$ and $\langle \rho_t \bar{\rho}_0 \rangle = \nu(t)$. The functions $\nu(t)$ and $g(t)$ are now arbitrary (apart from the condition $g(t) = 0$ for $t < 0$); hence they do not, in general, satisfy equation (55). Note that equation (78) provides a model for the dynamics of a confined Brownian particle in an out-of-equilibrium medium [34]–[36].

The dissipated power is given by

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 2\alpha \ \mathrm{Im} \ \dot{z}_t \bar{z}_t - 2\,\mathrm{Re} \int_{-\infty}^{\infty} \mathrm{d}t' \ g(t-t')\dot{z}_t \dot{\bar{z}}_{t'} + 2\,\mathrm{Re} \ \dot{z}_t \bar{\rho}_t = W_t - \widetilde{W}_t, \tag{79}$$

where as in the previous case $W_t = 2\alpha \, \mathrm{Im} \ \dot{z}_t \bar{z}_t$ is the power injected by the external force and $\widetilde{W}_t = 2\,\mathrm{Re} \int_{-\infty}^{\infty} \mathrm{d}t' \, g(t-t')\dot{z}_t \dot{\bar{z}}_{t'} - 2\,\mathrm{Re} \ \dot{z}_t \bar{\rho}_t$ is the power extracted by the bath.

For the harmonic model, σ_t^V is a boundary term and equation (41) gives

$$\mathcal{S}_\tau^{\mathrm{diss}} = -2\alpha \frac{\Delta\omega}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\omega_n |z_n|^2}{T(\omega_n)} = 2\alpha\mathrm{i} \int_{-\tau/2}^{\tau/2} \mathrm{d}t \int_{-\tau/2}^{\tau/2} \mathrm{d}t' \ T^{-1}(t-t')z_t \dot{\bar{z}}_{t'}. \tag{80}$$

Note that the last equality holds neglecting boundary terms.

Let us now compute $\phi_{\mathrm{diss}}(\lambda)$. The computation is straightforward following the strategy of section 3. In Fourier space, equation (78) reads

$$z_\omega = \frac{\rho_\omega}{-m\omega^2 + \kappa - \mathrm{i}\omega g(\omega)} = \frac{\rho_\omega}{D(\omega)}. \tag{81}$$

The probability distribution of ρ_ω is

$$\mathcal{P}[\rho_\omega] = \exp\left[-\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{|\rho_\omega|^2}{\nu(\omega)}\right]. \tag{82}$$

Thus (see equations (72) and (73)),

$$
\begin{aligned}
\langle \exp[-\lambda \mathcal{S}_\tau^{\mathrm{diss}}] \rangle &= \mathcal{N}^{-1} \int \mathrm{d}\rho_n \exp\left[-\frac{\Delta\omega}{2\pi} \sum_{n=-\infty}^{\infty} \left(\frac{|\rho_n|^2}{\nu(\omega_n)} - \frac{2\alpha\lambda\omega_n |\rho_n|^2}{T(\omega_n)|D(\omega_n)|^2} \right) \right] \\
&= \prod_{n=-\infty}^{\infty} \left[1 - \frac{2\alpha\lambda\omega_n \nu(\omega_n)}{T(\omega_n)|D(\omega_n)|^2} \right]^{-1},
\end{aligned}
\tag{83}
$$

and using the definition of $T(\omega)$ given by equation (18),

$$
\phi_{\mathrm{diss}}(\lambda) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \ln\left[1 - \frac{4\alpha\lambda\omega\,\mathrm{Re}\,g(\omega)}{|D(\omega)|^2} \right].
\tag{84}
$$

It is easy to prove that $|D(\omega)|^2 - 4\alpha\omega\,\mathrm{Re}\,g(\omega) = |D(-\omega)|^2$. Using now the same trick as was employed in equation (75), one shows that $\phi_{\mathrm{diss}}(\lambda) = \phi_{\mathrm{diss}}(1-\lambda)$.

An alternative definition of entropy production rate in which one assumes that it is proportional to the power injected by the external drive, $\sigma_t^\Theta = \Theta^{-1} W_t$, via a parameter Θ which has the dimension of a temperature, has often been used in the literature [21]–[23], [42]. With this definition, the total entropy production over a time τ is given by (neglecting boundary terms)

$$
\mathcal{S}_\tau^\Theta = \frac{2\alpha\mathrm{i}}{\Theta} \int_{-\tau/2}^{\tau/2} \mathrm{d}t\; z_t \dot{\bar{z}}_t = -2\alpha \frac{\Delta\omega}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\omega_n |z_n|^2}{\Theta},
\tag{85}
$$

i.e. $T(\omega)$ is replaced by a constant Θ that is taken as a free parameter that one adjusts in such a way that the PDF of $\mathcal{S}_\tau^\Theta$ is as a close as possible to verifying a fluctuation relation [21]–[23]. Replacing $T(\omega)$ with a constant Θ in equation (83), one obtains

$$
\phi_\Theta(\lambda) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \ln\left[1 - \frac{2\alpha\lambda\omega\nu(\omega)}{\Theta|D(\omega)|^2} \right].
\tag{86}
$$

However, it is not possible to find a value of Θ such that $\phi_\Theta(\lambda)$ satisfies the fluctuation theorem for the harmonic problem in contact with a generic bath. We shall show in section 4.2 that the use of a single parameter Θ constitutes a rather good approximation when the dynamics of the particle occurs on a single timescale.

In conclusion, the fluctuation theorem is satisfied when the entropy production rate is defined using the power injected by the external drive with the temperature of the environment defined as in (18).

In section 3.1.2 we also introduced a complex bath made of many equilibrated baths at different temperatures, eventually acting on different timescales. In appendix B we prove that, as expected, the PDF of $\mathcal{S}_\tau^{\mathrm{diss}}$ defined in equation (80) also verifies the fluctuation theorem in this case—while the PDF of $\mathcal{S}_\tau^\Theta$ does not. For such a multiple bath one can also consider the entropy production of the baths, defined as the power extracted by each bath divided by the corresponding temperature. This quantity is of interest if one can identify the different thermal baths with which the system is in contact; clearly, this is not possible in glassy systems where the effective temperature is self-generated. Nevertheless, the study of systems of particles coupled to many baths at different temperatures is of interest in the study of heat conduction. In appendix C we prove that the entropy production rate of the baths verifies the fluctuation theorem, at least for $|p| \leq 1$ (see also [40, 41]).

4. A driven particle in an anharmonic potential: numerical results

In this section we investigate numerically equation (78) for a particular choice of the non-equilibrium bath and in the presence of a linear and nonlinear interaction. In the linear case, we find that the numerical results reproduce the analytical results of the previous section. This finding confirms that the boundary terms we neglected in the analytical computation are indeed irrelevant. In the nonlinear case, we find that the fluctuation relation seems to be satisfied quite well for $\mathcal{S}_\tau^{\mathrm{diss}}$, although strictly speaking it only holds for $\mathcal{S}_\tau^{\mathrm{diss}} + \mathcal{S}_\tau^V$. We shall discuss the reason for this below.

We consider the simplest non-trivial case, in which a massless Brownian particle is coupled to two equilibrated baths: a white (or fast) bath at temperature T_{f} and a coloured (or slow) bath with exponential correlation at temperature T_{s}. This model has been studied in detail in [10] and is relevant for the description of glassy dynamics when the timescales of the two baths are well separated, as will be discussed in section 5. The equations of motion are given by equation (78) with $g(t) = g_{\mathrm{f}}(t) + g_{\mathrm{s}}(t)$, $g_{\mathrm{f}}(t) = \gamma_{\mathrm{f}}\delta(t)$ and $g_{\mathrm{s}}(t) = \theta(t)(\gamma_{\mathrm{s}}/\tau_{\mathrm{s}})\mathrm{e}^{-t/\tau_{\mathrm{s}}}$ or, equivalently, $g_{\mathrm{f}}(\omega) = \gamma_{\mathrm{f}}$ and $g_{\mathrm{s}}(\omega) = \gamma_{\mathrm{s}}/(1 - \mathrm{i}\omega\tau_{\mathrm{s}})$. We use a generic rotationally invariant potential $V(x,y) = \mathcal{V}[(x^2 + y^2)/2] = \mathcal{V}(|z|^2)$. The noise is the sum of a fast and a slow component. Then equation (78) becomes

$$\gamma_{\mathrm{f}}\dot{z}_t + \frac{\gamma_{\mathrm{s}}}{\tau_{\mathrm{s}}}\int_{-\infty}^t \mathrm{d}t' \, \mathrm{e}^{-(t-t')/\tau_{\mathrm{s}}}\dot{z}_{t'} = -z_t\mathcal{V}'(|z_t|^2) + \mathrm{i}\alpha z_t + \rho_t^{\mathrm{f}} + \rho_t^{\mathrm{s}}, \tag{87}$$

where $\langle\rho_t^{\mathrm{f}}\rho_{t'}^{\mathrm{f}}\rangle = 2\gamma_{\mathrm{f}}T_{\mathrm{f}}\delta(t - t')$, $\langle\rho_t^{\mathrm{s}}\rho_{t'}^{\mathrm{s}}\rangle = (T_{\mathrm{s}}\gamma_{\mathrm{s}}/\tau_{\mathrm{s}})\mathrm{e}^{-|t-t'|/\tau_{\mathrm{s}}}$ and $\mathcal{V}'(x)$ is the derivative of $\mathcal{V}(x)$ with respect to x. It is convenient to rewrite this equation as

$$\begin{aligned}\dot{u}_t &= -\frac{u_t - v_t}{\tau_{\mathrm{s}}} + \frac{\gamma_{\mathrm{s}}z_t}{\tau_{\mathrm{s}}^2}, \\ \gamma_{\mathrm{f}}\dot{z}_t &= -z_t\mathcal{V}'(|z_t|^2) + \mathrm{i}\alpha z_t + \rho_t^{\mathrm{f}} + u_t - \frac{\gamma_{\mathrm{s}}z_t}{\tau_{\mathrm{s}}},\end{aligned} \tag{88}$$

where we introduced the auxiliary variable u_t and v_t is a white noise with correlation $\langle v_t\bar{v}_{t'}\rangle = 2\gamma_{\mathrm{s}}T_{\mathrm{s}}\delta(t - t')$. The power injected by the external force is, as usual, $W_t = 2\alpha\,\mathrm{Im}\,\dot{z}_t\bar{z}_t$, while the power extracted by the two baths can be written as $\widetilde{W}_t^{\mathrm{f}} = 2\,\mathrm{Re}\,[\dot{z}_t\left(\gamma_{\mathrm{f}}\dot{\bar{z}}_t - \bar{\rho}_t^{\mathrm{f}}\right)]$ and $\widetilde{W}_t^{\mathrm{s}} = 2\,\mathrm{Re}\,[\dot{z}_t((\gamma_{\mathrm{s}}/\tau_{\mathrm{s}})\bar{z}_t - \bar{u}_t)]$.

For concreteness we focus on the potential $\mathcal{V}(|z|^2) = (g/2)|z|^4$ and compare with the results obtained for the harmonic case, $\mathcal{V}(|z|^2) = k|z|^2$. The simulation has been performed for $\alpha = 0.5$, $T_{\mathrm{f}} = 0.6$, $\gamma_{\mathrm{f}} = 1$, $T_{\mathrm{s}} = 2$, $\gamma_{\mathrm{s}} = 1$ and $\tau_{\mathrm{s}} = 1$; we set $k = 1$ in the linear case and $g = 1$ in the nonlinear one. The system (88) is numerically solved via a standard discretization of the equations with time step $\delta t = 0.01$; the noises are extracted using the routine GASDEV of the C numerical recipes [43].

We found that $\mathcal{S}_\tau^V$ and $\mathcal{S}_\tau^{\mathrm{diss}}$ are uncorrelated (within the precision of the numerical data). To the extent that this is the case, their PDFs can be studied separately. Unfortunately, the PDF of $\mathcal{S}_\tau^V$ is too noisy to allow for a verification of the fluctuation relation in the nonlinear case. This is probably due to the fact that while in the linear case $\mathcal{S}_\tau^V$ reduces to a boundary term[10], in the non-linear case 'spurious' boundary contributions might be difficult to eliminate [40]. Indeed, for the accessible values of τ, the variance of

---

[10] This is also observed in the simulation, because the average of $\mathcal{S}_\tau^V$ vanishes and the variance of $\mathcal{S}_\tau^V$ does not grow with τ.

$\mathcal{S}_\tau^V$ is much larger than its average (while the fluctuation relation predicts a variance of the order of σ_+^V). The large variance might be a finite-τ effect due to a boundary term with fluctuations contributing to those of $\mathcal{S}_\tau^V$ but not to the average. If this were the case, the fluctuation relation should hold for $|p| < 1$ and very large τ. However, the required values of τ might be so large as to render the fluctuation relation (FR) unobservable in practice; see [19, 41]. For this reason, we shall not discuss the data for $\mathcal{S}_\tau^V$. The validity of the FR for $\mathcal{S}_\tau^V$ (possibly minus a boundary term) in the nonlinear case remains an open question that should be addressed by future work.

4.1. Entropy production rate

Let us now discuss the behaviour of $\mathcal{S}_\tau^{\mathrm{diss}}$. The dissipative contribution to the entropy production rate—see equation (41)—is given by

$$\sigma_t^{\mathrm{diss}} = \int_{-\infty}^t \mathrm{d}t' \, T^{-1}(t - t') \, 2\alpha \, \mathrm{Im} \left[\dot{z}_t \bar{z}_{t'} + \dot{z}_{t'} \bar{z}_t \right]. \tag{89}$$

The inverse of the frequency dependent temperature, $1/T(\omega)$, is

$$\frac{1}{T(\omega)} = \frac{\gamma_{\mathrm{f}}(1 + \omega^2 \tau_{\mathrm{s}}^2) + \gamma_{\mathrm{s}}}{T_{\mathrm{f}} \gamma_{\mathrm{f}}(1 + \omega^2 \tau_{\mathrm{s}}^2) + T_{\mathrm{s}} \gamma_{\mathrm{s}}}; \tag{90}$$

see equation (61). Thus

$$T^{-1}(t) = \frac{1}{T_{\mathrm{f}}} \delta(t) + \frac{\gamma_{\mathrm{s}}}{T_{\mathrm{f}} \gamma_{\mathrm{f}} \tau_{\mathrm{s}}^2} \left(1 - \frac{T_{\mathrm{s}}}{T_{\mathrm{f}}} \right) \frac{\mathrm{e}^{-\Omega|t|}}{2\Omega}, \qquad \text{with } \Omega = \frac{1}{\tau_{\mathrm{s}}} \sqrt{\frac{T_{\mathrm{f}} \gamma_{\mathrm{f}} + T_{\mathrm{s}} \gamma_{\mathrm{s}}}{T_{\mathrm{f}} \gamma_{\mathrm{f}}}}, \tag{91}$$

and $T^{-1}(t)$ decays exponentially for large t. Note that, if the bath is in equilibrium, $T_{\mathrm{s}} = T_{\mathrm{f}} = T$, one has $T^{-1}(t) = \delta(t)/T$ and $\sigma_t^{\mathrm{diss}} = 2\alpha \, \mathrm{Im} \, \dot{z}_t \bar{z}_t / T = W_t / T$ as expected (recall that in our convention $\int_{-\infty}^t \mathrm{d}t' \, \delta(t - t') = \frac{1}{2}$).

The data for $\mathcal{S}_\tau^{\mathrm{diss}}$ are shown in figure 1. The large deviation function $\zeta_{\mathrm{diss}}(p)$ is reported in panel (a) for the harmonic and in panel (c) for the quartic potential. The average σ_+^{diss} is equal to 0.332 in the harmonic case and to 0.276 in the quartic case. The function $\zeta_{\mathrm{diss}}(p)$ converges fast to its asymptotic limit $\tau \to \infty$ (note that even the data for $\tau \sim 10$ are in quite good agreement with the analytic prediction for the harmonic case). The fluctuation theorem predicts $f(p) \equiv [\zeta_{\mathrm{diss}}(p) - \zeta_{\mathrm{diss}}(-p)]/\sigma_+^{\mathrm{diss}} = p$. The function $f(p)$ is reported in panel (b) for the harmonic and in panel (d) for the quartic potential. In the harmonic case the numerical data are compatible with the validity of the fluctuation theorem, as predicted analytically. Remarkably, the same happens in the quartic case for which we do not have an analytical prediction.

These results support the conjecture that, if $\mathcal{S}_\tau^{\mathrm{diss}}$ and $\mathcal{S}_\tau^V$ are uncorrelated, the PDF of $\mathcal{S}_\tau^{\mathrm{diss}}$ verifies the fluctuation theorem independently of the form of the potential $V(x, y)$.

4.2. Approximate entropy production rate

We also investigated numerically the fluctuations of the entropy production $\mathcal{S}_\tau^\Theta$ used in some numerical and experimental studies [21]–[23], [42] and that we discussed in section 3.2.2. For this model it is given by

$$\sigma_t^\Theta = \frac{W_t}{\Theta} = \frac{2\alpha}{\Theta} \mathrm{Im} \, \dot{z}_t \bar{z}_t. \tag{92}$$
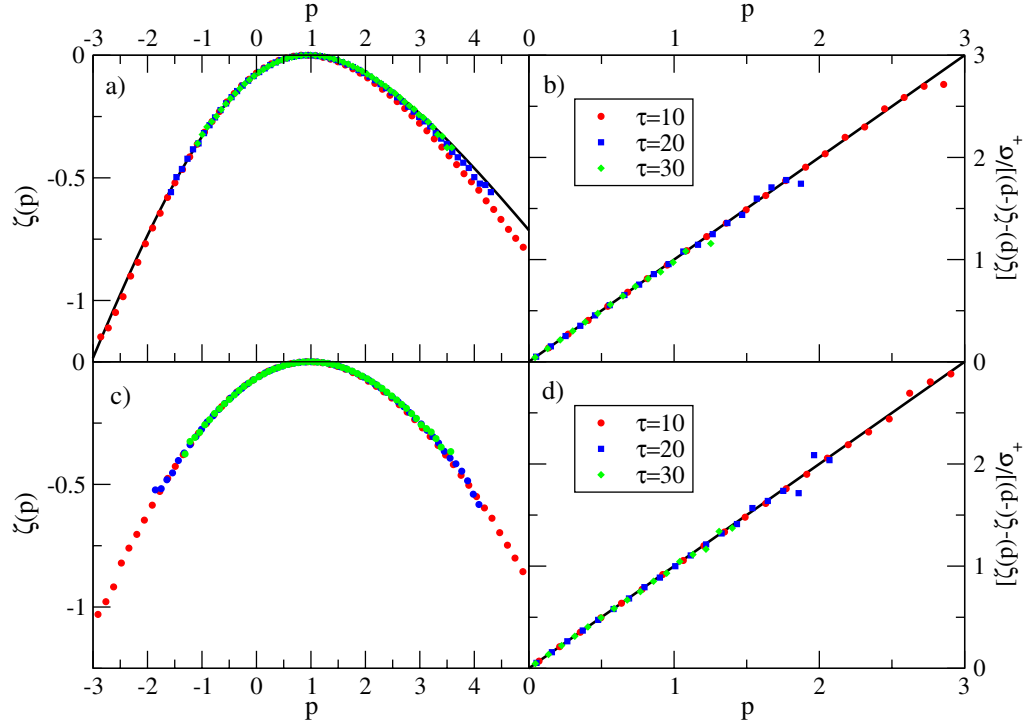
Figure 1. PDF of σ_t^{diss}. (a) The large deviation function for the harmonic potential at $\tau = 10, 20, 30$; the full line is the analytical result. (b) The function $f(p) \equiv [\zeta_{\text{diss}}(p) - \zeta_{\text{diss}}(-p)]/\sigma_+^{\text{diss}}$ for the harmonic potential: the fluctuation theorem predicts a straight line with slope 1, represented by a full line. (c) The large deviation function for the quartic potential at $\tau = 10, 20, 30$. (d) The function $f(p)$ for the quartic potential; in this case also the fluctuation theorem is well verified.

Rather arbitrarily we set $\Theta = T_{\text{f}}$ in the definition of σ_t^{Θ}. This reflects what is usually done in numerical simulations, where the dissipated power is divided by the 'kinetic' temperature, i.e. the temperature of the fast degrees of freedom. Note that the choice $\Theta = T_{\text{f}}$ does not affect the function $\zeta_{\Theta}(p)$ since the variable p is normalized (i.e., $\zeta_{\Theta}(p) \equiv \zeta(p)$ does not depend on Θ; see equation (1)) but it changes the average σ_+^{Θ} which is proportional to Θ^{-1}.

The data for σ_t^{Θ} are reported in figure 2. The harmonic case is shown in panels (a) and (b) while the anharmonic case is presented in panels (c) and (d). We have $\sigma_+^{T_{\text{f}}} = 0.455$ for the harmonic potential and $\sigma_+^{T_{\text{f}}} = 0.366$ for the quartic one. The numerical result for the large deviation function of σ_t^{Θ} agrees very well with the analytical prediction in the harmonic case but, as discussed in section 3.2.2, it does not verify the fluctuation theorem for $\Theta = T_{\text{f}}$, as one can clearly see from the right panels in figure 2.

Remarkably, in both the harmonic and anharmonic cases the function $f(p) \equiv [\zeta(p) - \zeta(-p)]/\sigma_+^{T_{\text{f}}}$ is approximately linear in p with a slope X such that $1 > X > T_{\text{f}}/T_{\text{s}}$, i.e. $\zeta(p) - \zeta(-p) \sim X \, p\sigma_+^{T_{\text{f}}}$. If $f(p) \sim Xp$, one can tune the value of Θ in order to obtain the fluctuation relation $\zeta(p) - \zeta(-p) = p\sigma_+^{\Theta}$, by simply choosing $\Theta = \Theta_{\text{eff}} = T_{\text{f}}/X$, thus defining a single 'effective temperature' $\Theta_{\text{eff}} \in [T_{\text{f}}, T_{\text{s}}]$. From the data reported in figure 2 we get a slope $X \sim 0.66$, which gives $\Theta_{\text{eff}} = T_{\text{f}}/X \sim 0.9$.
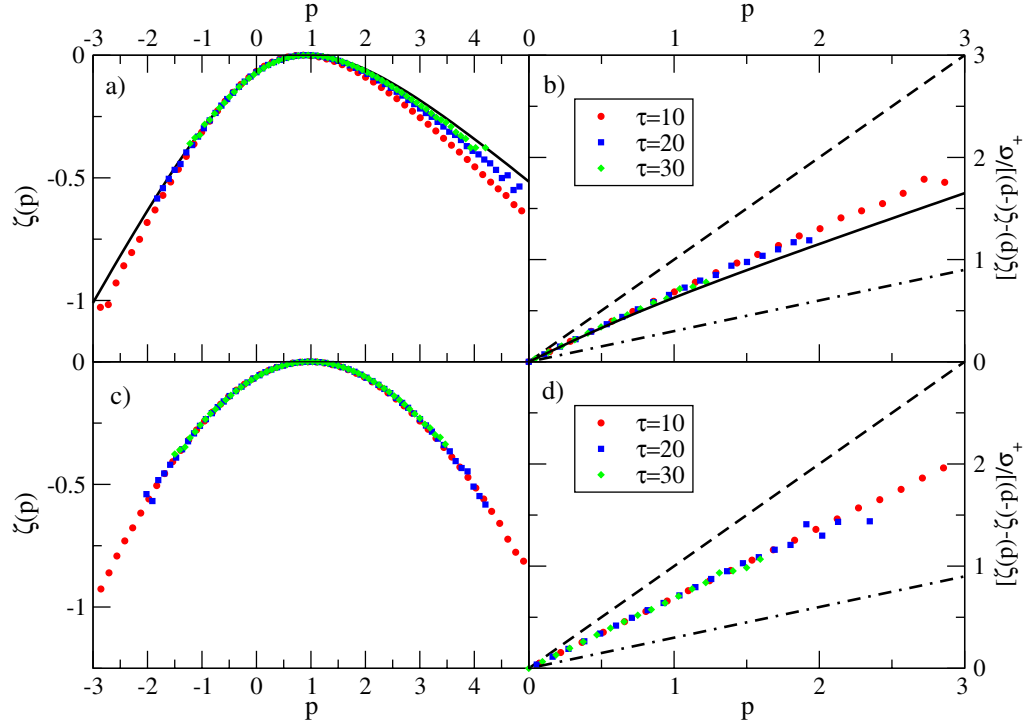
Figure 2. PDF of σ_t^Θ. (a) The large deviation function for the harmonic potential at $\tau = 10, 20, 30$; the full line is the analytical result. (b) The function $f(p) \equiv [\zeta(p) - \zeta(-p)]/\sigma_+^{T_f}$ for the harmonic potential. The full line is the analytical prediction, the dashed line is the fluctuation relation and the dot–dashed line has slope T_f/T_s. (c) The large deviation function for the quartic potential at $\tau = 10, 20, 30$. (d) The function $f(p)$ for the quartic potential; the dashed line is the fluctuation theorem and the dot–dashed line has slope T_f/T_s.

This behaviour reflects the one found in some recent experiments [21]–[23], [39] in situations in which the dynamics of the system happens essentially on a single timescale. This is the case also in our numerical simulation: in figure 3 we report the autocorrelation function $C(t) = \mathrm{Re}\langle z_t \bar{z}_0 \rangle$ of z_t (computed in appendix D) for the harmonic potential. The present simulation refers to the curve with $\tau_s = 1$, which clearly decays on a single timescale.

In figure 4 we report the parametric plot $\chi(C)$ (see sections 1 and 3) for the same set of parameters, but $\alpha = 0$. The integrated response is given by $\chi(t) = \int_0^t \mathrm{d}t'\, R(t')$ and $R(t)$ is computed in appendix D. We see that, for $\tau_s = 1$, the function $\chi(C)$ has a slope close to $-1/T_f$ at short times (corresponding to $\chi \sim 0$). For longer times, the slope moves continuously toward $-1/T_{\mathrm{eff}}$, with $T_{\mathrm{eff}} \sim 1.37$. This value of T_{eff} is of the order of $(\gamma_f T_f + \gamma_s T_s)/(\gamma_f + \gamma_s) = 1.3$, which means that on timescales of the order of the (unique) relaxation time the two baths behave like a single bath equilibrated at an intermediate temperature. This would be *exact* if the timescales of the two baths were exactly equal.

It is worth noting that in this situation we get $T_{\mathrm{eff}} \neq \Theta_{\mathrm{eff}}$; that is, the effective temperature that one would extract from the *approximate* fluctuation relation of figure 2 does not coincide with the effective temperature obtained from the $\chi(C)$ plot of figure 4.
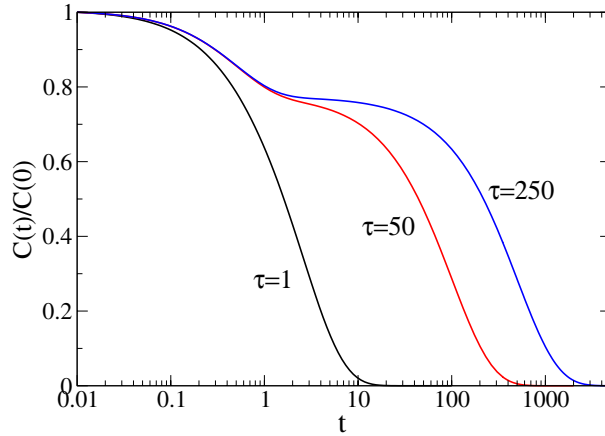
Figure 3. Normalized autocorrelation functions of z_t for the harmonic oscillator with $\alpha = 0.5$, $k = 1$, $T_f = 0.6$, $T_s = 2$, $\gamma_f = 1$, $\gamma_s = k\tau_s$ and $\tau_s = 1, 50, 250$.
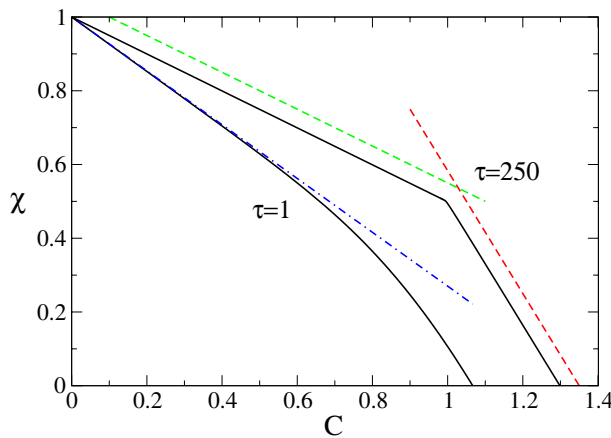

Figure 4. Parametric plot of the integrated response $\chi(t)$ as a function of the correlation function $C(t)$ for the harmonic oscillator with $\alpha = 0$, $k = 1$, $T_f = 0.6$, $T_s = 2$, $\gamma_f = 1$, $\gamma_s = k\tau_s$ and $\tau_s = 1$ (continuous curve) and $\tau_s = 250$ (broken curve). The dot–dashed line has slope $-1/1.37$; the dashed lines have slope $-1/T_s$ and $-1/T_f$.

In particular, we get $T_f < \Theta_{\text{eff}} < T_{\text{eff}}$: this relation is consistent with the results of [39] obtained from the numerical simulation of a sheared Lennard-Jones-like mixture, although the coincidence might be accidental.

4.3. Discussion

Let us summarize the results in this section. The numerical simulation of the non-linear problem confirms that the fluctuation theorem is satisfied *exactly* when the entropy production rate σ_t^{diss} is defined using the power injected by the external drive and the temperature in (18) is used.

In situations in which the dynamics of the system happens on a single timescale, a fitting parameter Θ can be introduced to obtain an *approximate* fluctuation relation

based on the entropy production rate W_t/Θ. However, Θ is not necessarily related to the effective temperature T_{eff} that enters the modified fluctuation-dissipation relation. In the systems considered so far [39] one finds $\Theta < T_{\text{eff}}$. However, this is just an approximation that fails in more generic non-equilibrium situations. In the next section we show that when the dynamics happens on different, well separated, timescales, it is impossible to find a single value of Θ such that $\sigma_t^\Theta = W_t/\Theta$ verifies the fluctuation relation.

5. Separation of timescales and driven glassy systems

In this section we discuss the application of our results to glassy systems. After presenting the general argument, we show explicitly that the dissipative entropy production satisfies the fluctuation relation for the p-spin spherical model. We finally discuss an adiabatic approximation that allows one to derive approximate results in the case of systems with well separated timescales.

5.1. Background

As discussed in section 1, in the study of mean field models for glassy dynamics [10, 33] and when using resummation techniques within a perturbative approach to microscopic glassy models with no disorder, effective equations of motion of the form of equation (10) are obtained:

$$\gamma \dot{\varphi}_t = -\mu(t)\ \varphi_t + \int_{-\infty}^t \mathrm{d}t'\ \Sigma(t,t')\ \varphi_{t'} + \rho_t + \alpha h[\varphi_t], \tag{93}$$

where ρ_t is a Gaussian noise such that $\langle \rho_t \rho_{t'} \rangle = D(t,t')$. The 'self-energy' $\Sigma(t,t')$ and the 'vertex' $D(t,t')$ depend on the interactions in the system through the correlation and response of the field φ. In the absence of drive ($\alpha = 0$) this equation has a dynamic transition at T_{d} separating a high temperature phase in which the dynamics rapidly equilibrates from a low temperature phase in which the dynamics is non-stationary and the fluctuation-dissipation relation is violated. This means that the time needed for the system to equilibrate is the longest timescale that is unreachable in the calculation (it already diverged with N).

In the case of a driven mean field system [25]–[27], the external force is also present in equation (93) and after a transient the system becomes stationary *for any temperature*, i.e. $\mu(t) \equiv \mu$, $\Sigma(t,t') = \Sigma(t-t')$ and $D(t,t') = D(t-t')$. The functions D and Σ depend on the strength α of the driving force. In this case, equation (93) resembles equations (78) and (87), and our results of section 3 apply. By analogy with equation (18), the *effective temperature* is defined in terms of Σ and D; see below.

As discussed in [25]–[29], for small α the system shows a completely different behaviour above and below T_{d}, reflecting the presence of a dynamical transition at $\alpha = 0$. Above T_{d}, the fluctuation-dissipation theorem holds in the limit of $\alpha \to 0$; the transport coefficient related to the driving force α approaches a constant value for $\alpha \to 0$ (the linear response holds close to equilibrium) and the system behaves like a 'Newtonian fluid'. Below T_{d}, the fluctuation-dissipation relation is violated also in the limit $\alpha \to 0$ and the transport coefficient diverges in this limit: the system is strongly nonlinear. For a wide class of systems—see section 1 and [33]—the relation between D and Σ takes a very simple form

in the limit $\alpha \to 0$: the effective temperature $T_{\text{eff}}(t - t')$ defined from the ratio between induced integrated response and correlation (see section 1) is given by the temperature of the bath T for small $|t - t'|$ and by a constant $T_{\text{eff}} > T$ for large $|t - t'|$. Thus, we expect that above T_{d} (and for $\alpha \sim 0$) the system behaves as if coupled to a single equilibrium bath (and the fluctuation theorem holds for the entropy production rate defined as $\sigma_t = \alpha h[\varphi_t]\dot\varphi_t/T$), while below T_{d} the system behaves as if coupled to two baths acting on different timescales and equilibrated at different temperatures.

As already remarked in the introduction, these single-spin equations of motion are valid for times τ that are finite with respect to the size N. For these times the system behaves like independent spins moving in a harmonic potential in contact with a non-equilibrium environment. Thus, for $\mathcal{S}_\tau$ with τ in this regime the results of section 3 apply and the correct definition of the entropy production rate is given by equation (80), i.e. by the power injected by the external force alone, divided by the frequency dependent effective temperature.

5.2. The spherical p spin

The (modified) fluctuation relation can be checked explicitly for the p-spin spherical model. This model realizes explicitly the behaviour described above. The asymptotic dynamics in the low temperature phase occurs in a region of phase space that is called the *threshold* and it is far from the equilibrium states [3] since times that grow with N are needed to reach them.

The effective equations of motion for the driven spherical p spin [25, 27] are

$$\partial_t C(t, t') = -\mu(t)C(t, t') + \int \mathrm{d}t'' \, \Sigma(t, t'')C(t'', t') + \int \mathrm{d}t'' \, D(t, t'')R(t', t''),$$

$$\partial_t R(t, t') = -\mu(t)R(t, t') + \int \mathrm{d}t'' \, \Sigma(t, t'')R(t'', t') + \delta(t - t'), \tag{94}$$

$$\mu(t) = T + \int \mathrm{d}t' \, [D(t, t')R(t, t') + \Sigma(t, t')C(t, t')],$$

with the vertex and self-energy

$$D = \frac{p}{2}C^{p-1} + \alpha^2 \frac{k}{2}C^{k-1} \equiv D_0 + \alpha^2 D_1,$$
$$\Sigma = \tfrac{1}{2}p(p-1)RC^{p-2} = RD_0'(C), \tag{95}$$

respectively. The dynamics can also be described with a 'single-spin' Langevin equation of the form

$$\dot\varphi(t) = -\mu(t)\varphi(t) + \int \mathrm{d}t' \, \Sigma(t, t')\varphi(t') + \rho(t),$$
$$\langle \rho(t)\rho(t') \rangle = 2T\delta(t - t') + D(t, t'). \tag{96}$$

Note that Σ and D_0 verify the detailed balance condition. From the expressions (95), one can rewrite equation (96), in the stationary case, in the following way:

$$\dot\varphi(t) = -\mu\varphi(t) + \int_{-\infty}^{\infty} \mathrm{d}t' \, \Sigma(t - t')\varphi(t') + \rho(t) + \alpha h(t),$$
$$\langle \rho(t)\rho(t') \rangle = 2T\delta(t - t') + D_0(t - t'), \tag{97}$$
$$\langle h(t)h(t') \rangle = D_1(t - t'),$$

where $\rho(t)$ and $h(t)$ are two uncorrelated Gaussian variables. Note that Σ and D_0 still depend implicitly on α as one has to solve the self-consistency equations (94) for C and R and substitute the result in Σ and D_0.

In the absence of drive and interpreting Σ and D as the response and correlation of a bath, its frequency dependent effective temperature is

$$T(\omega) = \frac{\nu(\omega)}{2 \operatorname{Re} g(\omega)} = \frac{2T + D_0(\omega)}{2 \operatorname{Re}[1 + \Sigma(\omega)/(\mathrm{i}\omega)]}. \tag{98}$$

Given that Σ and D depend on R and C, if one finds that R and C are related by the fluctuation-dissipation theorem, $R(t) = -\beta\theta(t)\dot{C}(t)$, from the relation $\Sigma = RD_0'(C)$ it follows that $T(\omega) \equiv T$ and the bath is in equilibrium, as expected. If R and C do not verify the fluctuation-dissipation theorem, $T(\omega) \neq T$.

Switching on the external drive one can compute its dissipated power

$$W(t) = \alpha h(t)\dot{\varphi}(t) \tag{99}$$

and its average. One finds

$$\langle W(t)\rangle = \alpha\langle h(t)\dot{\varphi}(t)\rangle = \alpha \left\langle h(t) \int_{-\infty}^{\infty} \mathrm{d}t' \, \dot{R}(t-t')[\rho(t') + \alpha h(t')]\right\rangle$$

$$= \langle W\rangle = \alpha^2 \int_0^{\infty} \mathrm{d}t \, \dot{R}(t)D_1(t) = \alpha^2 \frac{k}{2}\int_0^{\infty} \mathrm{d}t \, \dot{R}(t)C^{k-1}(t), \tag{100}$$

consistently with the result of [27] where the average of the injected power was explicitly computed for this model. One can prove that the entropy production $\mathcal{S}_\tau^{\mathrm{diss}}$ obtained from the rate

$$\sigma_t^{\mathrm{diss}} = \alpha \int_{-\infty}^{t} \mathrm{d}t' \, T^{-1}(t-t')[h(t)\dot{\varphi}(t') + h(t')\dot{\varphi}(t)], \tag{101}$$

verifies the fluctuation relation. Indeed, first rewriting the dissipative entropy production as

$$\mathcal{S}_\tau^{\mathrm{diss}} = \alpha\frac{\Delta\omega}{2\pi} \sum_{n=-\infty}^{\infty} \frac{h_n \mathrm{i}\omega_n \bar{\varphi}_n}{T(\omega_n)}, \tag{102}$$

and using the solution for φ_n, in the generic notation of the previous section, one finds

$$\phi_{\mathrm{diss}}(\lambda) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \ln\left[1 + 4\alpha^2\lambda(1-\lambda)\frac{\mu(\omega)\omega^2[\operatorname{Re} g(\omega)]^2}{\nu(\omega)|D(\omega)|^2}\right], \tag{103}$$

where ν, μ and g are the Fourier transforms of the ρ–ρ correlator, the h–h correlator and the time-integrated Σ, respectively. It is easy to check that $\phi_{\mathrm{diss}}(\lambda) = \phi_{\mathrm{diss}}(1-\lambda)$ and the fluctuation relation is then verified.

Once again, we showed that the dissipative contribution to the entropy production satisfies the fluctuation relation with a modified temperature. Note however that this solvable example is non-trivial for at least two reasons. It is a clearly anharmonic problem since Σ and D are themselves functions of C and R. The single spin φ is coupled to an equilibrated bath at temperature T *and* a self-generated 'bath' at a different temperature. The temperature entering the fluctuation relation involves both, through the definition (98). This temperature is the one that one would observe by measuring the fluctuation-dissipation relation for the variable φ.

29

5.3. The adiabatic approximation

When a simple system is coupled to a complex bath with two (or more) timescales these are induced into the dynamics of the system. When the timescales are well separated, an adiabatic treatment is possible in which one separates the dynamic variables into terms that evolve on different timescales (dictated by the baths) and are otherwise approximately constant.

In this subsection we use an adiabatic approach [10] to treat simple problems coupled to baths that evolve on different scales. The motivation for studying such problems is that they resemble glassy systems although in the latter the separation of timescales is self-generated.

We study the PDF of $\mathcal{S}_\tau^{\mathrm{diss}}$ and $\mathcal{S}_\tau^\Theta$. The former satisfies the fluctuation theorem (at least in the harmonic case since corrections due to $\mathcal{S}_\tau^V$ might appear in non-harmonic problems). We check that the adiabatic approximation does not spoil this feature. The latter, instead, does not satisfy the fluctuation theorem in general. However, it is interesting to understand under which conditions it satisfies the fluctuation theorem approximately. Indeed, in numerical simulations and experiments it has been customary to measure the dissipated power $W_t = \alpha h[\varphi_t]\dot\varphi_t$ and then define the entropy production rate as $\sigma_t = W_t/T$ (where T is the temperature of the fast bath). This corresponds to the definition of σ_t^Θ, given in equation (85) for the harmonic oscillator. We show that when the particle is coupled to a bath that evolves on well separated timescales, σ_t^Θ *does not* satisfy the fluctuation theorem.

5.3.1. The harmonic model coupled to two baths. Let us consider again the Langevin equation (87) with $\mathcal{V}(|z|^2) = k|z|^2$. In this case, the correlation functions can be calculated explicitly; see appendix D. In figure 3 we report the autocorrelation functions, $C(t) = \langle z_t \bar z_0 \rangle$, for $\alpha = 0.5$, $k = 1$, $T_{\mathrm{f}} = 0.6$, $T_{\mathrm{s}} = 2$, $\gamma_{\mathrm{f}} = 1$, $\gamma_{\mathrm{s}} = k\tau_{\mathrm{s}}$ and different values of τ_{s}. Clearly, for $k\tau_{\mathrm{s}} = \gamma_{\mathrm{s}} \gg \gamma_{\mathrm{f}}$ two very different timescales—related to the timescales of the two baths—are present. From the plot of figure 4 one sees that in the case $k\tau_{\mathrm{s}} = 250 \gg \gamma_{\mathrm{f}}$ the function $\chi(C)$ is a broken line with slope $-1/T_{\mathrm{f}}$ at large C (short times) and $-1/T_{\mathrm{s}}$ for small C (large times).

We want to show that, in this situation, the variable z_t can be written as the sum of two quasi-independent contributions. Using the construction introduced in [10] we rewrite the equation of motion (87) as

$$
\begin{aligned}
\gamma_{\mathrm{f}}\dot z_t &= -(k + \gamma_{\mathrm{s}}/\tau_{\mathrm{s}})z_t + \mathrm{i}\alpha z_t + \rho_t^{\mathrm{f}} + h_t, \\
h_t &= -\frac{\gamma_{\mathrm{s}}}{(\tau_{\mathrm{s}})^2}\int_{-\infty}^t \mathrm{d}t'\ \mathrm{e}^{-(t-t')/\tau_{\mathrm{s}}}z_{t'} + \rho_t^{\mathrm{s}}.
\end{aligned}
\tag{104}
$$

The variable h_t is 'slow'; if we consider it as a constant in the first equation, the variable z_t will fluctuate around the equilibrium position $z_h = h/(k + \gamma_{\mathrm{s}}/\tau_{\mathrm{s}} - \mathrm{i}\alpha) \equiv H$. The latter will—slowly—evolve according to the second equation in (104), in which we can approximate $z_{t'} \sim H_{t'}$. Defining the—fast—displacement of z_t with respect to H_t, $w_t \equiv z_t - H_t$, we obtain the following equations for (w_t, H_t):

$$
\begin{aligned}
\gamma_{\mathrm{f}}\dot w_t &= -(k + \gamma_{\mathrm{s}}/\tau_{\mathrm{s}})w_t + \mathrm{i}\alpha w_t + \rho_t^{\mathrm{f}}, \\
\frac{\gamma_{\mathrm{s}}}{\tau_{\mathrm{s}}}\int_{-\infty}^t \mathrm{d}t'\ \mathrm{e}^{-(t-t')/\tau_{\mathrm{s}}}\dot H_{t'} &= -kH_t + \mathrm{i}\alpha H_t + \rho_t^{\mathrm{s}}.
\end{aligned}
\tag{105}
$$

In this approximation, $z_t = H_t + w_t$ is the sum of two contributions: w_t is a 'fast' variable which evolves according to a Langevin equation with the fast bath only and a renormalized harmonic constant $k + \gamma_s/\tau_s$, while H_t is a 'slow' variable which evolves according to a Langevin equation where the slow bath only appears. In both equations the driving force α is present; thus we expect both H_t and w_t to contribute to the dissipation. Note that w_t and H_t are completely uncorrelated in this approximation.

5.3.2. The 'potential' entropy production rate σ_t^V. In the adiabatic approximation the term $\mathcal{S}_\tau^V$ in equation (41) should become a boundary term. Indeed, the function $T^{-1}(t)$, in the adiabatic approximation, becomes

$$T^{-1}(t) = \frac{1}{T_f}\delta(t) + T_s^{-1}(t), \tag{106}$$

where the function $T_s^{-1}(t)$ is 'slow'; see e.g. equation (91). Inserting this expression in σ_t^V, the first term gives a total derivative. The second term gives

$$\int_{-\infty}^t \mathrm{d}t'\, T_s^{-1}(t - t') \left[\dot{r}_\alpha(t)\frac{\partial V}{\partial r_\alpha}[\vec{r}(t')] + \dot{r}_\alpha(t')\frac{\partial V}{\partial r_\alpha}[\vec{r}(t)] \right]. \tag{107}$$

Due to the convolution with the 'slow' function $T_s^{-1}(t)$, the fast components of r are irrelevant in the integral, while for the slow ones it is reasonable to replace $\dot{r}_\alpha(t)$ with $\dot{r}_\alpha(t')$ on the scale τ_s over which $T_s^{-1}(t)$ decays. Thus one obtains a total derivative times the integral of $T_s^{-1}(t)$ which is a finite constant. Obviously this is not a rigorous proof and should be checked numerically in concrete cases.

5.3.3. The 'dissipative' entropy production rate σ_t^{diss}. The entropy production rate defined in equations (80) and (89) can be rewritten in terms of H_t and w_t. Recalling that $T^{-1}(t)$ is defined by equation (91) one obtains (the details of the calculation are reported in appendix E)

$$\sigma_t^{\mathrm{diss}} \sim 2\alpha\,\mathrm{Im}\left[\frac{\dot{w}_t\bar{w}_t}{T_f} + \frac{\dot{H}_t\bar{H}_t}{T_s} \right], \tag{108}$$

neglecting terms that vanish when σ_t^{diss} is integrated over time intervals of the order of τ_s. This is exactly the entropy production expected for two independent systems.

To check that this approximation works well, let us introduce a 'power spectrum' $\sigma(\omega)\mathrm{d}\omega$ as the contribution coming from frequencies $[\omega, \omega + \mathrm{d}\omega]$ to the average entropy production rate, $\sigma_+^{\mathrm{diss}} = \int_0^\infty \mathrm{d}\omega\, \sigma(\omega)$. From equation (84) we get

$$\sigma_+^{\mathrm{diss}} = \frac{\mathrm{d}\phi_{\mathrm{diss}}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = -\int_{-\infty}^\infty \frac{\mathrm{d}\omega}{2\pi}\frac{2\alpha\omega\nu(\omega)}{|D(\omega)|^2}$$
$$\sigma(\omega) = \frac{1}{2\pi}\left[-\frac{2\alpha\omega\nu(\omega)}{|D(\omega)|^2} + \frac{2\alpha\omega\nu(-\omega)}{|D(-\omega)|^2} \right] = \frac{\alpha\omega\nu(\omega)}{\pi}\left[\frac{1}{|D(-\omega)|^2} - \frac{1}{|D(\omega)|^2} \right]. \tag{109}$$

Substituting the expressions for $\nu(\omega)$ and $D(\omega)$ appropriate for equation (87) we get the power spectrum $\sigma(\omega)$ as a function of ω which is reported in figure 5 as a full line. The contributions of w_t and H_t, $\sigma_w(\omega)$ and $\sigma_H(\omega)$, are obtained by inserting in equation (109) the expression for $\nu(\omega)$ and $D(\omega)$ obtained from the two equations (105). They are
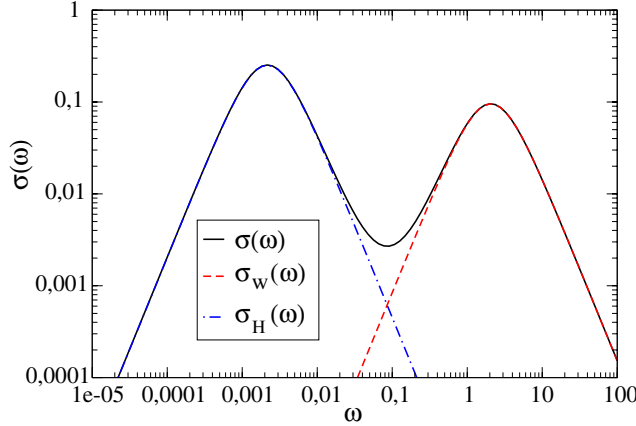
Figure 5. Power spectrum of the entropy production rate (full line) as a function of the frequency for the harmonic oscillator with $\alpha = 0.5$, $k = 1$, $T_\text{f} = 0.6$, $T_\text{s} = 2$, $\gamma_\text{f} = 1$, $\gamma_\text{s} = k\tau_\text{s}$ and $\tau_\text{s} = 250$. The dot–dashed line is the 'slow' contribution of H_t; the dashed line is the 'fast' contribution of w_t.

reported as dashed and dot–dashed lines in figure 5. We can conclude that, for $k\tau_\text{s} \gg \gamma_\text{f}$, the adiabatic approximation holds and $\sigma_t^\text{diss} = \sigma_t^w + \sigma_t^H$, with $\sigma_t^w = 2\alpha \operatorname{Im} \dot{w}_t \bar{w}_t / T_\text{f}$ and $\sigma_t^H = 2\alpha \operatorname{Im} \dot{H}_t \bar{H}_t / T_\text{s}$, and the two contributions are independent. Note that the average dissipation due to H is much larger than the one due to w. Finally, we can write

$$\phi_\text{diss}(\lambda) = \phi^w(\lambda) + \phi^H(\lambda), \qquad \phi^{w,H}(\lambda) = -\lim_{\tau \to \infty} \tau^{-1} \ln \langle \exp[-\lambda \sigma^{w,H}] \rangle. \tag{110}$$

Both $\phi^w(\lambda)$ and $\phi^H(\lambda)$ verify the fluctuation theorem, as the two equations of motion (105) are particular instances of the general case discussed in section 3. The function $\zeta_\text{diss}(p)$ is the Legendre transform of $\phi_\text{diss}(\lambda)$ and will verify the fluctuation theorem.

5.3.4. The 'approximate' entropy production rate σ_t^Θ. In the same approximation, σ_t^Θ is given, for $\Theta = T_\text{f}$, by

$$\sigma_t^{T_\text{f}} = 2\alpha \operatorname{Im} \left[\frac{\dot{w}_t \bar{w}_t + \dot{H}_t \bar{H}_t}{T_\text{f}} \right] = \sigma_t^w + \frac{T_\text{s}}{T_\text{f}} \sigma_t^H, \qquad \text{and}$$

$$\phi_{T_\text{f}}(\lambda) = \phi^w(\lambda) + \phi^H(\lambda T_\text{s}/T_\text{f}); \tag{111}$$

the contribution of H_t is weighted with the 'wrong' temperature, i.e. the temperature of the fast degrees of freedom. Indeed, as we have already discussed, $\phi_\Theta(\lambda)$ does not verify the fluctuation theorem. The function $f(p) \equiv [\zeta_\Theta(p) - \zeta_\Theta(-p)]/\sigma_+^\Theta$ (obtained from equation (86)) is reported in figure 6. As already discussed in section 4, when the timescales of the two baths are of the same order, $k\tau_\text{s} \sim \gamma_\text{f}$, the two baths act like a single bath at temperature $\Theta \in [T_\text{f}, T_\text{s}]$ and the function $f(p)$ is approximately linear in p with slope $X \in [T_\text{f}/T_\text{s}, 1]$. When the timescales are well separated, $k\tau_\text{s} \gg \gamma_\text{f}$, the adiabatic approximation holds; and one finds that $f(p)$ has slope of ~ 1 for small p and T_f/T_s for large p (see figure 6).

The results for $k\tau_\text{s} \sim \gamma_\text{f}$ are consistent with the ones in [39] where only the situation in which the two timescales are not well separated could be investigated. Indeed, when the
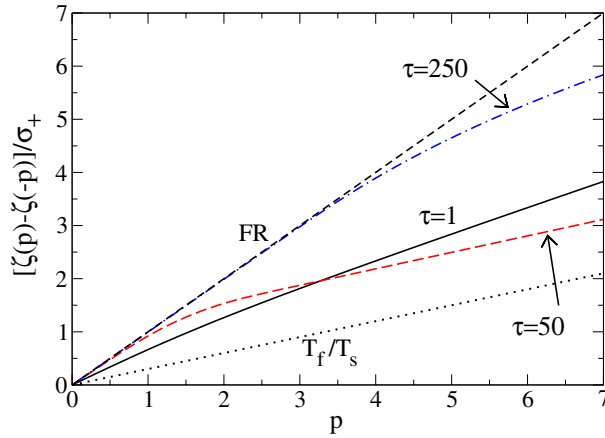
Figure 6. The function $[\zeta_\Theta(p) - \zeta_\Theta(-p)]/\sigma_+^\Theta$ for the harmonic oscillator with $\alpha = 0.5$, $k = 1$, $\Theta = T_{\rm f} = 0.6$, $T_{\rm s} = 2$, $\gamma_{\rm f} = 1$, $\gamma_{\rm s} = k\tau_{\rm s}$ and $\tau_{\rm s} = 1, 50, 250$. The dashed line (FR) is a straight line with slope 1; the dotted line has slope $T_{\rm f}/T_{\rm s} = 0.3$.

dynamics becomes very slow the observation of large negative fluctuations of the entropy production requires a huge amount of computational time and the function $f(p)$ can be calculated only in a narrow range of p around $p = 0$. Note that the value of p at which the slope of $f(p)$ crosses over from 1 to $T_{\rm f}/T_{\rm s}$ depends on the values of the parameters α, $\gamma_{\rm f}$, $\gamma_{\rm s}$, $\tau_{\rm s}$ etc and can be of the order of 5, while in numerical simulations one can usually reach values of—at most—$p \sim 3$. Thus, the observation of curves like the one reported in figure 6 in numerical simulations of glassy systems is a very difficult task.

6. Green–Kubo relations

It was proven in [31, 32] that the fluctuation theorem implies, in the equilibrium limit $(\sigma_+ \to 0)$, the Green–Kubo relation for transport coefficients. This is a particular form of the fluctuation-dissipation theorem. In this section we discuss how one links the modified fluctuation theorem—in which we replaced the external bath temperature by the (frequency dependent) effective temperature of the unperturbed system—to the modification of the fluctuation-dissipation relation.

6.1. General derivation

Let us recall briefly how the Green–Kubo relation can be obtained from the fluctuation theorem. Suppose that we apply a (constant) driving force E to a system in equilibrium. This generates a corresponding flux J_t (e.g. if E is an electric field, J_t is the electric current) such that, close to equilibrium, the dissipated power can be written as $W_t = EJ_t$. The entropy production rate is then

$$\sigma_t = \frac{W_t}{T} = \frac{EJ_t}{T}. \tag{112}$$

33

The fluctuation theorem can be written as $\phi(\lambda) = \phi(1 - \lambda)$ where $\phi(\lambda)$ is defined in equation (25). The derivatives of $\phi(\lambda)$ are the moments of $\mathcal{S}_\tau$, i.e.

$$\phi_k \equiv \left.\frac{\mathrm{d}^k \phi}{\mathrm{d}\lambda^k}\right|_{\lambda=0} = (-1)^{k-1} \lim_{\tau\to\infty} \tau^{-1} \langle \mathcal{S}_\tau^k \rangle_c, \tag{113}$$

where $\langle A^2 \rangle_c = \langle A^2 \rangle - \langle A \rangle^2$ and so on. Thus, $\phi_k \sim E^k$ (for $k \geq 2$), and close to equilibrium ($E \sim 0$) $\phi(\lambda)$ is well approximated by a second-order polynomial (corresponding to a Gaussian PDF),

$$\phi(\lambda) \sim \phi_0 + \phi_1 \lambda + \frac{\phi_2}{2} \lambda^2. \tag{114}$$

Then the fluctuation relation, $\phi(\lambda) = \phi(1 - \lambda)$, implies $\phi_2 = -2\phi_1$; from equation (113) and using time-translation invariance,

$$\phi_2 = -2\phi_1 \Rightarrow \sigma_+ = \int_0^\infty \mathrm{d}t \, \langle \sigma_t \sigma_0 \rangle_c. \tag{115}$$

Substituting $\sigma_t = E J_t / T$ one obtains

$$\langle J \rangle_E = \frac{E}{T} \int_0^\infty \mathrm{d}t \, \langle J_t J_0 \rangle_{E=0} + \mathrm{o}(E^2), \tag{116}$$

that is to say, the Green–Kubo relation.

Note that even out of equilibrium one can define a flux $\mathcal{J}_t$ using σ_t as a 'Lagrangian'—see [42]:

$$\mathcal{J}_t = \frac{\partial \sigma_t}{\partial E}. \tag{117}$$

Close to equilibrium σ_t is given by equation (112) and $\mathcal{J}_t = J_t / T$. If, in the absence of a drive, the system has a non-trivial effective temperature, the entropy production rate should be defined as in equations (80) and (89). Then the flux $\mathcal{J}_t$ is given by

$$\mathcal{J}_t = \frac{\partial \sigma_t^{\mathrm{diss}}}{\partial \alpha} = 4 \, \mathrm{Im} \int_{-\infty}^t \mathrm{d}t' \, T^{-1}(t - t') \dot{z}_t \bar{z}_{t'} = 2 \int_{-\infty}^t \mathrm{d}t' \, T^{-1}(t - t')[\dot{y}_t x_{t'} - \dot{x}_t y_{t'}]. \tag{118}$$

The fluctuation theorem for σ^{diss} implies then a Green–Kubo relation for $\mathcal{J}_t$:

$$\langle \mathcal{J} \rangle_\alpha = \alpha \int_0^\infty \mathrm{d}t \, \langle \mathcal{J}_t \mathcal{J}_0 \rangle_{\alpha=0} + \mathrm{o}(\alpha^2). \tag{119}$$

The physical meaning of the latter relation becomes clear if one writes the flux $\mathcal{J}_t$ in the adiabatic approximation discussed in the previous section; from equation (108),

$$\mathcal{J}_t = 2 \, \mathrm{Im} \left[\frac{\dot{w}_t \bar{w}_t}{T_{\mathrm{f}}} + \frac{\dot{H}_t \bar{H}_t}{T_{\mathrm{s}}} \right] = \frac{J_t^w}{T_{\mathrm{f}}} + \frac{J_t^H}{T_{\mathrm{s}}}, \tag{120}$$

and equation (119) becomes

$$\frac{\langle J^w \rangle_\alpha}{T_{\mathrm{f}}} + \frac{\langle J^H \rangle_\alpha}{T_{\mathrm{s}}} = \frac{\alpha}{T_{\mathrm{f}}^2} \int_0^\infty \mathrm{d}t \, \langle J_t^w J_0^w \rangle_{\alpha=0} + \frac{\alpha}{T_{\mathrm{s}}^2} \int_0^\infty \mathrm{d}t \, \langle J_t^H J_0^H \rangle_{\alpha=0} + \mathrm{o}(\alpha^2). \tag{121}$$

Indeed, in the adiabatic approximation the Green–Kubo relation holds separately for J_t^w (with temperature $T_{\rm f}$) and for J_t^H (with temperature $T_{\rm s}$). Equation (119) encodes the two contributions and holds even when the adiabatic approximation does not apply and the contributions of the 'fast' and of the 'slow' modes is not well separated.

Note that the 'classical' Green–Kubo relation involves the total flux $J_t = J_t^w + J_t^H$. For the latter one has, in the adiabatic approximation,

$$
\begin{aligned}
\langle J_t \rangle_\alpha = \langle J_t^w \rangle_\alpha + \langle J_t^H \rangle_\alpha &= \frac{\alpha}{T_{\rm f}} \int_0^\infty \mathrm{d}t \; \langle J_t^w J_0^w \rangle_{\alpha=0} + \frac{\alpha}{T_{\rm s}} \int_0^\infty \mathrm{d}t \; \langle J_t^H J_0^H \rangle_{\alpha=0} \\
&= \alpha \int_0^\infty \mathrm{d}t \left[\frac{\langle J_t^w J_0^w \rangle_{\alpha=0}}{T_{\rm f}} + \frac{\langle J_t^H J_0^H \rangle_{\alpha=0}}{T_{\rm s}} \right] \sim \alpha \int_0^\infty \mathrm{d}t \frac{1}{T_{\rm eff}(t)} \langle J_t J_0 \rangle_{\alpha=0}.
\end{aligned}
\tag{122}
$$

The latter relation is the generalization of the Green–Kubo formula that comes from the generalized fluctuation-dissipation relation discussed in the sections 1 and 3. It is closely related, but not equivalent, to equation (119).

6.2. The Green–Kubo relation for driven glassy systems

Equations (119) and (122) cannot be applied straightforwardly to driven glassy systems, as for these systems the correlation function $\langle J_t J_0 \rangle_\alpha$ is not stationary at $\alpha = 0$ at low temperatures. Indeed, the relaxation time of the latter grows very fast as $\alpha \to 0$ and at some point falls outside the experimentally accessible range: the system will not be able to reach stationarity on the experimental timescales and will start to *age* indefinitely.

However, let us consider again the equation of motion (10) for $\alpha \neq 0$, where we assume that a stationary state is reached:

$$
\gamma \dot{\varphi}_t = -\mu_\alpha \, \varphi_t + \int_{-\infty}^t \mathrm{d}t' \, \Sigma_\alpha(t-t') \, \varphi_{t'} + \rho_t + \alpha h[\varphi_t], \qquad \langle \rho_t \rho_{t'} \rangle = D_\alpha(t-t').
\tag{123}
$$

The functions $\Sigma_\alpha(t-t')$ and $D_\alpha(t-t')$ depend (strongly) on α; indeed, as the term explicitly proportional to α is a small perturbation for $\alpha \sim 0$, the main contribution to the α dependence of the dynamics of φ_t will come from the α dependence of Σ_α and D_α. If we compare the latter equation with equation (87), we see that setting $\alpha = 0$ in equation (87) is equivalent to setting $\alpha = 0$ *without* changing the functions Σ and D in equation (123). This will not affect too much the correlation function $\langle J_t J_0 \rangle_\alpha$ if α is small. Finally, we can write, for small α,

$$
\langle J_t \rangle_\alpha \sim \alpha \int_0^\infty \mathrm{d}t \frac{1}{T_{\rm eff}(t)} \langle J_t J_0 \rangle_\alpha,
\tag{124}
$$

even if the limit $\alpha \to 0$ is not well defined. An analogous relation will be obtained from equation (119) (which is equivalent to the fluctuation theorem in the Gaussian approximation) within the same approximation. The latter relations can be tested in numerical simulations as well as in experiments.

7. Slow periodic drive and effective temperature

In this section we discuss a means to measure an effective temperature associated with slow timescales of a non-equilibrium system by using the modification of the fluctuation theorem.

A lesson we learn from the previous calculations (see e.g. figure 5) is that the work done at large frequencies is overwhelmingly larger than that done at very low frequencies— precisely what we wish to observe in order to detect effective temperatures. One way out of this is to choose a perturbation that does little work at high frequencies: a periodically time dependent force that *derives from a potential* $\cos(\Omega t)\tilde{V}(r)$, with $1/\Omega$ of the order of the timescale of the slow bath τ_s. Let us show this for a one-dimensional system, the generalization being straightforward.

7.1. General derivation

Let us consider a single degree of freedom r moving in a time independent potential $V(r)$ and subject to a periodically time dependent field $\cos(\Omega t)\tilde{V}(r)$, and in contact with a 'fast' and a 'slow' bath with friction kernel, thermal noise and temperature (ρ^f, g_f, T_f) and (ρ^s, g_s, T_s), respectively:

$$
m\ddot{r}(t) + \int_{-\infty}^{t} \mathrm{d}t' \, g_f(t - t')\dot{r}(t') + \int_{-\infty}^{t} \mathrm{d}t' \, g_s(t - t')\dot{r}(t')
$$
$$
= -\frac{\partial V}{\partial r}[r(t)] + \rho^f(t) + \rho^s(t) - \cos(\Omega t)\frac{\partial \tilde{V}}{\partial r}[r(t)]. \tag{125}
$$

The timescale of the time dependent field $1/\Omega$ is of the same order as that of the 'slow' bath. The work in an interval of time $(0, \tau)$ done by the time dependent potential is

$$
W_\tau = -\int_0^\tau \cos(\Omega t')\frac{\partial \tilde{V}}{\partial r}\dot{r} \, \mathrm{d}t' = -\tilde{V}(\tau) + \tilde{V}(0) + \Omega \int_0^\tau \sin(\Omega t') \, \tilde{V} \, \mathrm{d}t'. \tag{126}
$$

Only the last term grows with the number of cycles, so for long times we can neglect the first two. Now, integrating (125) by parts, we obtain

$$
m\ddot{r}(t) = -\int_{-\infty}^{t} \mathrm{d}t' \, g_f(t - t')\dot{r}(t') - \frac{\partial V}{\partial r}[r(t)] + \rho^f + h(t) - \hat{h}(t)\frac{\partial \tilde{V}}{\partial r}[r(t)] \tag{127}
$$

$$
h(t) \equiv -\int_{-\infty}^{t} \mathrm{d}t' \, g_s(t - t')r(t') + \rho^s(t), \tag{128}
$$

where $\hat{h}(t) = \cos(\Omega t)$. In the adiabatic limit when both the timescales of the slow bath and the period $1/\Omega$ of the potential $\tilde{V}$ are large, $h(t)$ and $\hat{h}(t)$ are quasi-static. Hence, r has a fast evolution given by equation (127) with h and $\hat{h}$ fixed and it reaches a distribution [10]

$$
P(r/h, \hat{h}) = \frac{\mathrm{e}^{-\beta_f\left(V + \hat{h}\tilde{V} + g_f(0)(r^2/2) - hr\right)}}{\int \mathrm{d}r \, \mathrm{e}^{-\beta_f\left(V + \hat{h}\tilde{V} + g_f(0)(r^2/2) - hr\right)}}. \tag{129}
$$

The denominator defines $Z(h, \hat{h})$ and $F(h, \hat{h}) \equiv -\beta_f^{-1} \ln Z(h, \hat{h})$. Note that $F(h, \hat{h}(t))$ is periodically time dependent through $\hat{h}$. The approximate evolution of h is now given by equation (128) with the replacement of r in the friction term by its average $\partial F(h, \hat{h})/\partial h$ with respect to the fast evolution:

$$
h(t) = \int_{-\infty}^{t} \mathrm{d}t' \, g_s(t - t')\frac{\partial F(h, \hat{h})}{\partial h}(t') + \rho^s(t). \tag{130}
$$

Equation (130) is in fact a generalized Langevin equation for a system coupled to a (slow) bath of temperature T_s. Indeed, it can be shown [10] to be equivalent to a set of degrees of freedom y_i evolving according to the ordinary Langevin equation:

$$\left[m_j \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \gamma_j \frac{\mathrm{d}}{\mathrm{d}t} + \Omega_j \right] y_j = \xi_j(t) - \frac{\partial F\left(\sum_j A_j y_j \right)}{\partial y_j} \tag{131}$$

with $\langle \xi_i(t)\xi_j(t') \rangle = 2T_s \gamma_j \delta_{ij} \delta(t-t')$, provided that the Fourier transforms $g_s(\omega)$ and $\nu_s(\omega)$ of the friction kernel and noise autocorrelation can be written as

$$g(\omega) = \sum_j \frac{A_j^2}{m_j(\omega - \omega_j^+)(\omega - \omega_j^-)}$$

$$\nu(\omega) = 2T_s \sum_j \frac{\gamma_j A_j^2}{m_j^2(\omega - \omega_j^+)(\omega - \omega_j^-)(\omega + \omega_j^+)(\omega + \omega_j^-)} \tag{132}$$

where $\omega_j^\pm$ are the roots of $-m_j \omega^2 + \mathrm{i}\gamma_j \omega + \Omega_j = 0$.

Within the approximation that also led to (130), the average of $\tilde{V}(r)$ over a time window Δ that is long compared to the short timescale but sufficiently slow that we can consider that h and $\hat{h}$ are constant is

$$\int_t^{t+\Delta} \tilde{V}(r(t'))\, \mathrm{d}t' \sim \Delta \int \mathrm{d}r\, P(r/h, \hat{h})\, \tilde{V}(r) = \Delta \frac{\partial F(h, \hat{h})}{\partial \hat{h}} \tag{133}$$

so that we obtain

$$W_\tau \sim \Omega \int_0^\tau \sin(\Omega t') \frac{\partial F(h, \hat{h}(t'))}{\partial \hat{h}}\, \mathrm{d}t' = -\int_0^\tau \frac{\partial F(h, \hat{h}(t'))}{\partial t'}\, \mathrm{d}t' \tag{134}$$

which tells us that for long time intervals the work done by the original time dependent potential $\tilde{V}$ is indeed the same as the work done by the time dependent effective potential F in (130).

The fluctuation theorem then holds for the distribution of this work, with a single temperature T_s. We conclude that the distribution of work due to a slow perturbation satisfies the fluctuation theorem with only the slow temperature, and can hence be used experimentally to detect it.

7.2. Experimental realization

The simplest application of the above general result is obtained considering $\tilde{V}(r) = \tilde{h}r$ and $V(r) = kr^2$. Then, grouping together the two noises into a single one with friction $g = g_f + g_s$ and correlation $\nu = T_f \nu_f + T_s \nu_s$ as described in section 3, equation (125) simply becomes

$$m\ddot{r}(t) + \int_{-\infty}^t \mathrm{d}t'\, g(t-t')\dot{r}(t') = -kr(t) + \rho(t) + \tilde{h}\cos(\Omega t). \tag{135}$$

This equation describes for instance the motion of a Brownian particle moving in an out-of-equilibrium environment and trapped by a harmonic potential whose centre oscillates at frequency Ω. A concrete experimental realization of this setting has already been considered in [34]: silica beads of $\sim 2\,\mu$m diameter were dispersed in a solution of laponite

(a particular clay of ~ 30 nm diameter particle size) and water. The laponite suspension forms a glass for large enough concentration of clay and provides the non-equilibrium environment. The silica beads are Brownian particles diffusing in such an environment. They can be trapped by optical tweezing, and the centre of the trap can oscillate with respect to the sample if the latter is oscillated through a piezoelectric stage. In [34] the mobility and diffusion of tracer particles were measured, providing an estimate of $T_{\text{eff}}(\Omega)$. Here we propose to measure the work done by the trap on the tracers. Indeed, the work dissipated in $(0, \tau)$ is linear in $r(t)$ so it should be possible to measure it simply through the measurement of $r(t)$:

$$W_\tau = \Omega \tilde{h} \int_0^\tau \mathrm{d}t' \sin(\Omega t') r(t'); \tag{136}$$

note that, as W_τ is linear in $r(t)$, it is a Gaussian variable. With a simple calculation one finds

$$\lim_{\tau \to \infty} \frac{\langle (W_\tau - \langle W_\tau \rangle)^2 \rangle}{2 \langle W_\tau \rangle} = \frac{\nu(\Omega)}{2 \operatorname{Re} g(\Omega)} = T_{\text{eff}}(\Omega). \tag{137}$$

This means that the (Gaussian) PDF of $\mathcal{S}_\tau^{\text{diss}} = W_\tau / T_{\text{eff}}(\Omega)$ satisfies the fluctuation relation. If the two baths are modelled as in section 4 with $k\gamma_{\text{s}} = \tau_{\text{s}} \gg \gamma_{\text{f}}$, one has $T_{\text{eff}}(\Omega) = T_{\text{s}}$ for $\Omega \tau_{\text{s}} < 1$ (see equation (90)). The measurement of the distribution of the work (136) allows for the measurement of T_{s}. Note that other experimental settings described by the same equations should exist.

8. Conclusions

We studied the extension of the fluctuation relation to open stochastic systems that are not able to equilibrate with their environments.

We used the simplest example to hand to test several generalized fluctuation formulae: a Brownian particle in a confining potential coupled to non-trivial external baths with different timescales and temperatures. Independently of the form of the potential energies, due to the coupling to the complex environment, the particle is not able to equilibrate. Its relaxational dynamics is characterized by an *effective temperature*, defined via the modification of the fluctuation-dissipation relation between spontaneous and induced fluctuations [9]. When no separation of timescales can be identified in the bath, the effective temperature is a non-trivial function of the two times involved. Instead, when the bath evolves in different timescales each characterized by a value of a temperature, the two-time dependent effective temperature is a piecewise function that actually takes only these values, each one characterizing the dynamics of the particle in a regime of times.

Several authors discussed the possibility of introducing the effective temperature in the fluctuation theorem to extend its domain of applicability to glassy models driven by external forces [12]–[15]. After summarizing our results we shall discuss how they compare to the proposals and findings in these papers.

8.1. Summary of results

We here examined carefully different definitions of the entropy production rate that are not equivalent when the *effective temperature* is not trivially equal to the ambient temperature. We found that:

(1) The PDF of the 'dissipative' entropy production $\mathcal{S}_\tau^{\text{diss}}$ that involves the *frequency dependent* temperature,

$$\mathcal{S}_\tau^{\text{diss}} = \frac{\Delta\omega}{2\pi} \sum_{n=-\infty}^{\infty} \frac{-\mathrm{i}\omega_n r_\alpha(\omega_n) h_\alpha(\omega_n)}{T(\omega_n)} = \int_{-\tau/2}^{\tau/2} \mathrm{d}t \int_{-\tau/2}^{\tau/2} \mathrm{d}t'\, T^{-1}(t-t')\dot{r}_\alpha(t)h_\alpha[\vec{r}(t')],$$

(138)

with $T^{-1}(t)$ the Fourier transform of $1/T(\omega)$, the effective temperature of the relaxing system—see equation (18)—verifies the fluctuation theorem exactly for harmonic system. It also holds for general systems connected to baths with different temperatures acting on widely separated scales.

(2) For nonlinear systems in contact with non-equilibrium baths acting on overlapping timescales an additional term $\mathcal{S}_\tau^V$—see equation (41)—has to be included in the entropy production to make the fluctuation relation hold strictly. Our numerical simulations suggest that, surprisingly enough, the effect of this extra 'internal' term is even then quite small.

(3) The PDF of $\mathcal{S}_\tau^\Theta$ with $\sigma_t^\Theta = W_t/\Theta$, W_t the *power dissipated by the external force* and Θ a free parameter with the dimensions of a temperature *does not* satisfy the fluctuation theorem in general for any choice of Θ. The large deviation function, $\zeta_\Theta(p)$, still shows some interesting features revealing the existence of an effective temperature. When the bath has, say, two components acting on different timescales and with different temperatures, the function $[\zeta_\Theta(p)-\zeta_\Theta(-p)]/\sigma_+^\Theta$ may have different slopes corresponding to these two temperatures, one at small p and the other at large p. The separation of timescales of the bath translates into a separation of scales in the function $[\zeta_\Theta(p) - \zeta_\Theta(-p)]/\sigma_+^\Theta$. When the timescales of the baths are not separated, and one records the large deviation function for not too large values of p only, the fluctuation theorem is verified approximately if Θ is suitably chosen. Note that the value of Θ found in this way is not equal to the effective temperature T_{eff} that enters the modified fluctuation-dissipation relation. Instead, when the timescales are well separated, the two scales in the large deviation function are clearly visible and a single fitting parameter is not sufficient to make the fluctuation theorem hold.

(4) If two timescales are present in the dynamics of a system and the applied perturbation is periodic with frequency $\Omega < 1/\tau_{\text{s}}$, τ_{s} being the largest relaxation time, the PDF of the power dissipated over a (large) number of cycles verifies the fluctuation relation with temperature $T_{\text{s}} = T_{\text{eff}}(\Omega)$. This is probably the easiest way of detecting the effective temperature by means of the fluctuation relation.

These results should apply to driven glassy systems as discussed in section 5 and are indeed consistent with recent numerical simulations [39]. Models like the one discussed here have been recently investigated [10], [34]–[36] in an effort to describe the dynamics of Brownian particles in complex media such as glasses and granular matter. Brownian particles are often used as probes in order to study the properties of the medium (e.g. in dynamic light scattering or diffusing wave spectroscopy experiments). Moreover, confining potentials for Brownian particles can be generated using laser beams [38] and experiments on the fluctuations of the power dissipated in such systems are currently being performed [34, 44].

8.2. Temperatures

It is important to summarize the different definitions of effective temperature that we considered and the relations between them. We defined the effective temperature in the frequency domain in equation (18) as a property of the bath which can also be measured from the ratio between correlation and response functions in the frequency domain. As we discussed above, *the same* effective temperature enters the correct definition of the entropy production rate *in the frequency domain*; see equation (138). Thus, experiments working in the frequency domain should observe the same effective temperature from the fluctuation-dissipation relation and the fluctuation relation.

In the time domain the situation is slightly more complicated. On the one hand, the effective temperature obtained from the fluctuation-dissipation relation *in the time domain*, defined for example by equation (9), is *not* the Fourier transform of $T(\omega)$. A convolution with the correlation function is involved in the relation between $T(\omega)$ and $T_{\text{eff}}(t)$. On the other hand, the effective temperature $T^{-1}(t)$ entering the entropy production is exactly the Fourier transform of $1/T(\omega)$; see again equation (138). This can give rise to ambiguities when working in the time domain.

Most of these ambiguities disappear as long as the timescales in the problem are well separated. In this case, on each timescale a well defined effective temperature can be identified, and this temperature enters both the fluctuation-dissipation relation and the fluctuation relation: see e.g. the curve for $\tau = 250$ in figure 4 and the expression for $\mathcal{S}_\tau^{\text{diss}}$ in the adiabatic approximation, equation (108). This is essentially related to the validity of the adiabatic approximation discussed in section 5.3.

The difference is relevant when the timescales of the two baths are not well separated, and a single effective temperature cannot be identified; see the curve for $\tau = 1$ in figure 4. In this case, we found that the fluctuation relation holds with—approximately—a single fitting parameter Θ but this temperature *is not clearly related* to the fluctuation-dissipation temperature *in the time domain*. This is indeed what is observed in numerical simulations on Lennard-Jones systems [39].

Let us remark again that, when applying these results to real glassy systems in finite dimensions, one should take care of the possibility that the effective temperature has some space fluctuations due to the heterogeneity of the dynamics [37]. The extension of our results to such a situation is left for future work.

8.3. Discussion

Several proposals for introducing the effective temperature into extensions of the fluctuation theorem have recently appeared in the literature. Let us confront them here with our results.

Sellitto studied the fluctuations of entropy production in a driven lattice gas with reversible kinetic constraints [12]. When coupling this system to an external particle reservoir with chemical potential μ, a dynamic crossover from a fluid to a glassy phase is found around μ_c. The glassy non-equilibrium phase is characterized by a violation of the fluctuation-dissipation theorem in which the parametric relation between the global integrated response and displacement yields a line with slope μ_{eff} [45].

One drives this (possibly already out-of-equilibrium) system by coupling two adjacent layers of the three-dimensional periodic cube to particle reservoirs at different chemical

potentials, μ_+ and μ_-. The former is allowed to assume a value corresponding to the glassy phase, $\mu_+ > \mu_c$, while μ_- is always below μ_c. The results of the Monte Carlo simulation are consistent with a generalized form of the fluctuation-dissipation theorem:

$$\mathcal{S}_\tau = J_\tau(\mu_{\text{eff}} - \mu_-), \tag{139}$$

where $\mathcal{S}_\tau$ is the entropy production, J_τ is the particle current in the direction of the externally imposed chemical potential gradient averaged over a time interval of duration τ; μ_{eff} is an effective chemical potential and μ_- is the chemical potential of one of the layers. When the chemical potentials of the two reservoirs are in the fluid phase, $\mu_{\text{eff}} = \mu_+$ and the usual fluctuation relation holds. Instead, when μ_+ is in the glassy phase, Sellitto found that equation (139) holds with μ_{eff} taking the value appearing in the violation of the fluctuation-dissipation theorem in the ageing regime of the *undriven* glassy system at μ_+.

The formula (139) differs from the ones we found to describe the oscillator problem in that in our case, when translating from temperature to chemical potential, the full time dependent $\mu(t)$ enters. Strictly, we believe that this improved definition should also apply to the lattice gas model. However in the case studied by Sellitto the fast dynamics is an 'intra-cage' dynamics that probably does not contribute to the current. This is a case in which the perturbation does not produce dissipation at high frequency, so the difference arising from $\mu(t) \neq \mu_{\text{eff}}$ should be tiny in this case (see section 7).

More recently, Crisanti and Ritort [14] found that the probability distribution function of the fluctuations of heat exchanges, Q, between an ageing random orthogonal model in its 'activated regime' (a long time regime in which the energy density decays as a logarithm of time) and the heat bath is rather well described by a stationary Gaussian part and a waiting time dependent exponential tail towards small values of Q. Assuming that these events are of two types ('stimulated' and 'spontaneous') they proposed to fit the ratio between the PDF of positive and negative 'spontaneous' Q s in the form of a fluctuation theorem, i.e. as proportional to $e^{-2Q/\lambda}$, and relate λ to the effective temperature of the fluctuation-dissipation relation. They found good agreement. Crisanti, Ritort and Picco are currently performing simulations to test this hypothesis for Lennard-Jones mixtures [46].

Another development is an attempt to generalize the situation considered by Crooks. He considered a problem that *starts from equilibrium in zero field* and evolves according to some stochastic dynamic rule in the presence of an arbitrary applied field [47] and found that the ratio between the probability of a trajectory and its time-reversed counterpart is given by $e^{-\beta \int_0^{t_{\max}} dt\, h(t)\dot{O}(t)}$ with $h(t)$ the time dependent external field that couples linearly to the observable O. For simplicity, let us focus on $O = \phi$ with ϕ a scalar field characterizing the system. In [13] the extension of this relation to the initial non-equilibrium 'glassy' case was conjectured. Separating the external fields h and ϕ into their fast and slow components [48], $h = h_{\text{f}} + h_{\text{s}}$ and $\phi = \phi_{\text{f}} + \phi_{\text{s}}$, one then proposes that the PDFs of the trajectories of the slow components satisfy a relation similar to Crooks' one with the temperature replaced by the effective temperature (for a glassy non-equilibrium system with two correlation scales [3]). This is indeed very similar to what we have done in this paper.

Finally, let us mention the work of Sasa [15] where he introduces an effective temperature in his definition of entropy production for the Kuramoto–Sivashinsky equation.

Acknowledgments

Appendix A. Dirichlet boundary conditions for the white bath

A second possibility for calculating the functional integral in equation (68) is imposing Dirichlet boundary conditions, $z(-\tau/2) = z(\tau/2) = 0$. However, in this case it is possible to calculate $\phi(\lambda)$ only for $m = 0$. The distribution of z_t is obtained by substituting $\rho_\omega = D(\omega)z_\omega$ in equation (72):

$$
\begin{aligned}
\mathcal{P}[z_t] &= \exp\left[-\frac{1}{2\gamma T}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\, z_\omega |D(\omega)|^2 \bar{z}_\omega\right]\\
&= \exp\left[-\frac{1}{2\gamma T}\int_{-\infty}^{\infty}\mathrm{d}t\, z_t\left(k^2 + \alpha^2 - 2\mathrm{i}\alpha\gamma\frac{\mathrm{d}}{\mathrm{d}t} - \gamma^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)\bar{z}_t\right].
\end{aligned}
\tag{A.1}
$$

From equation (68),

$$
\begin{aligned}
\langle\exp[-\lambda\mathcal{S}_\tau]\rangle = \mathcal{N}^{-1}\int\mathrm{d}z_t\,\exp\Bigg[&-\frac{1}{2\gamma T}\int_{-\infty}^{\infty}\mathrm{d}t\, z_t\big(k^2 + \alpha^2\\
&- 2\mathrm{i}\alpha\gamma[1 - 2\lambda\chi_\tau(t)]\frac{\mathrm{d}}{\mathrm{d}t} - \gamma^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big)\bar{z}_t\Bigg],
\end{aligned}
\tag{A.2}
$$

where $\chi_\tau(t)$ is the characteristic function of $t \in [-\tau/2, \tau/2]$. At the leading order in τ, as the correlation function of z_t decays exponentially on a timescale $\tau_0 = \gamma k^{-1}$, we can integrate out the portion of the trajectory that is outside the interval $[-\tau/2, \tau/2]$ both in the numerator and the denominator, and we simply obtain

$$
\begin{aligned}
\langle\exp[-\lambda\mathcal{S}_\tau]\rangle = \mathcal{N}^{-1}\int\mathrm{d}z_t\,\exp\Bigg[&-\frac{1}{2\gamma T}\int_{-\tau/2}^{\tau/2}\mathrm{d}t\, z_t\bigg(k^2 + \alpha^2\\
&- 2\mathrm{i}\alpha\gamma(1 - 2\lambda)\frac{\mathrm{d}}{\mathrm{d}t} - \gamma^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg)\bar{z}_t\Bigg].
\end{aligned}
\tag{A.3}
$$

We have then to find the eigenvalues of the operator appearing in the integral. This corresponds to finding the solution of the equation

$$
J\bar{z}_t = \left(k^2 + \alpha^2 - 2\mathrm{i}\alpha\gamma(1 - 2\lambda)\frac{\mathrm{d}}{\mathrm{d}t} - \gamma^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)\bar{z}_t = E\bar{z}_t
\tag{A.4}
$$

with boundary conditions $\bar{z}(\tau/2) = \bar{z}(-\tau/2) = 0$. Note that the operator J is Hermitian and thus the eigenvalues are real; they are given by the following expression:

$$
E_n(\lambda) = k^2 + 4\alpha^2\lambda(1 - \lambda) + \gamma^2\frac{\pi^2 n^2}{\tau^2}
\tag{A.5}
$$

with $n = 0, 1, \ldots$. For each n the integration is performed over one complex variable and we get

$$\langle \exp[-\lambda \mathcal{S}_\tau] \rangle = \mathcal{N}^{-1} \int \mathrm{d}z_t \exp\left[-\frac{1}{2\gamma T} \int_{-\tau/2}^{\tau/2} \mathrm{d}t \; z_t J \bar{z}_t \right] = \prod_{n=0}^{\infty} \frac{E_n(0)}{E_n(\lambda)} \quad \text{(A.6)}$$

recalling that the constant $\mathcal{N}$ is simply the numerator calculated in $\lambda = 0$. Finally we obtain, defining $\omega = n\pi/\tau$,

$$\phi(\lambda) = \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{n=0}^{\infty} \ln \frac{E_n(\lambda)}{E_n(0)} = \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \ln\left[1 + \frac{4\alpha^2 \lambda(1-\lambda)}{k^2 + \gamma^2 \omega^2} \right]. \quad \text{(A.7)}$$

The latter expression obviously verifies the fluctuation theorem. Moreover, in the $m = 0$ case equation (74) is equal to equation (A.7), as one can check using suitable changes of variable in the integral. In this simple case, $\zeta(p)$ can be computed exactly. Starting from equation (A.7) one has

$$\phi'(\lambda) = \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \frac{4\alpha^2(1-2\lambda)}{\gamma^2 \omega^2 + k^2 + 4\alpha^2 \lambda(1-\lambda)} = \frac{2\alpha^2(1-2\lambda)}{\gamma \sqrt{k^2 + 4\alpha^2 \lambda(1-\lambda)}} \quad \text{(A.8)}$$

and, recalling that $\phi(0) = 0$,

$$\phi(\lambda) = \int_0^\lambda \mathrm{d}\mu \; \phi'(\mu) = \gamma^{-1}[\sqrt{k^2 + 4\alpha^2 \lambda(1-\lambda)} - k]. \quad \text{(A.9)}$$

The function $\zeta(p)$ is defined by

$$\zeta(p) = \min_\lambda[\lambda p \sigma_+ - \phi(\lambda)] = \lambda^* p \sigma_+ - \phi(\lambda^*), \quad \text{(A.10)}$$

where $\sigma_+ = 2\alpha^2/(\gamma k)$ and λ^* is defined by $\phi'(\lambda^*) = p\sigma_+$; hence,

$$p = \frac{k(1-2\lambda^*)}{\sqrt{k^2 + 4\alpha^2 \lambda^*(1-\lambda^*)}} \Rightarrow \lambda^* = \frac{1}{2}\left[1 - p\sqrt{\frac{\alpha^2 + k^2}{\alpha^2 p^2 + k^2}} \right], \quad \text{(A.11)}$$

and finally

$$\zeta(p) = \gamma^{-1}\left\{ k + \frac{\alpha^2 p}{k}\left[1 - p\sqrt{\frac{\alpha^2 + k^2}{\alpha^2 p^2 + k^2}} \right] - k\sqrt{\frac{\alpha^2 + k^2}{\alpha^2 p^2 + k^2}} \right\}. \quad \text{(A.12)}$$

From the latter expression it is easy to verify that

$$\zeta(p) - \zeta(-p) = \frac{2\alpha^2 p}{k} = p\sigma_+, \quad \text{(A.13)}$$

as indicated by the FT. Defining $\tau_0 = \gamma/k$, the relaxation time of the correlation function of z_t, and $\sigma_0 = \sigma_+ \tau_0/2 = \alpha^2/k^2$, the (adimensional) entropy production over a time $\tau_0/2$, we obtain

$$\zeta(p) = \tau_0^{-1}\left[1 + p\sigma_0 - \sqrt{(1+\sigma_0)(1+p^2\sigma_0)} \right]. \quad \text{(A.14)}$$

Appendix B. Fluctuation theorem for many equilibrium baths at different temperatures

We compute the function $\phi(\lambda)$ in the case in which the driven oscillator is coupled to N coloured baths with generic memory functions and in equilibrium at different temperatures. The violation of the fluctuation-dissipation theorem for the relaxing particle in such an environment was discussed in section 3.1.2. As discussed there, the equations are mathematically equivalent to the ones discussed in section 3; thus the strategy as well as many details of the calculation are the same as in this section.

The equations of motion are

$$m\ddot{z}_t + \sum_{i=1}^{N} \int_{-\infty}^{\infty} \mathrm{d}s\ g_i(t-s)\dot{z}_s = -\kappa z_t + \sum_{i=1}^{N} \rho_{it}, \qquad (B.1)$$

with $\kappa = k - \mathrm{i}\alpha$. The thermal noises satisfy

$$\begin{aligned}
\langle \rho_{it}\rho_{j0} \rangle = \langle \bar{\rho}_{it}\bar{\rho}_{j0} \rangle &= 0, \\
\langle \rho_{it}\bar{\rho}_{j0} \rangle &= \delta_{ij} T_i \nu_i(t).
\end{aligned} \qquad (B.2)$$

By causality, the functions $g_i(t)$ must vanish for $t < 0$. As the baths are in equilibrium at temperature T_i, the functions $\nu_i(t)$ and $g_i(t)$ are related by equation (55):

$$\begin{aligned}
\nu_i(t) &= T_i[g_i(t) + g_i(-t)] = T_i g_i(|t|), \\
T_i g_i(t) &= \theta(t)\nu_i(t).
\end{aligned} \qquad (B.3)$$

In the frequency domain, equation (B.1) becomes

$$z_\omega = \frac{\sum_i \rho_{i\omega}}{-m\omega^2 + \kappa - \mathrm{i}\omega \sum_i g_i(\omega)} \equiv \frac{\sum_i \rho_{i\omega}}{D(\omega)}, \qquad (B.4)$$

where $D(\omega) = -m\omega^2 + \kappa - \mathrm{i}\omega \sum_i g_i(\omega)$.

The dissipated power is given by

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 2\alpha\ \mathrm{Im}\ \dot{z}_t\bar{z}_t - 2\,\mathrm{Re}\sum_i \int_{-\infty}^{\infty} \mathrm{d}s\ g_i(t-s)\dot{z}_t\dot{\bar{z}}_s + 2\,\mathrm{Re}\sum_i \dot{z}_t\bar{\rho}_{it} = W_t - \sum_i \widetilde{W}_{it}, \quad (B.5)$$

where, as in the previous cases, $W_t = 2\alpha\,\mathrm{Im}\,\dot{z}_t\bar{z}_t$ is the power injected by the external force and $\widetilde{W}_{it} = 2\,\mathrm{Re}\int_{-\infty}^{\infty} \mathrm{d}s\ g_i(t-s)\dot{z}_t\dot{\bar{z}}_s - 2\,\mathrm{Re}\,\dot{z}_t\bar{\rho}_{it}$ is the power extracted by the ith bath.

The first definition of the entropy production rate, equation (85), gives (in the following, we will always make the substitution $(\Delta\omega/2\pi)\sum_{n=-\infty}^{\infty} \to \int_{-\infty}^{\infty}(\mathrm{d}\omega/2\pi)$ as the error is $O(1)$ for $\tau \to \infty$; see section 3)

$$\mathcal{S}_\tau^\Theta = -\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{2\alpha\omega|z_\omega|^2}{\Theta}. \qquad (B.6)$$

Making the substitution $z_\omega = \sum_i \rho_{i\omega}/D(\omega)$, we obtain

$$\langle \exp[-\lambda\mathcal{S}_\tau^\Theta] \rangle = \mathcal{N}^{-1} \int \mathrm{d}\rho_{i\omega}\ \exp\left[-\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \sum_{ij} \rho_{i\omega} A_{ij}^\lambda(\omega)\bar{\rho}_{j\omega} \right], \qquad (B.7)$$

where $A^\lambda(\omega)$ is an $N \times N$ *real* matrix whose elements are given by

$$A_{ij}^\lambda(\omega) = \frac{\delta_{ij}}{T_i \nu_i(\omega)} - \frac{\lambda}{|D(\omega)|^2} \frac{2\alpha\omega}{\Theta}. \tag{B.8}$$

Then,

$$\phi_\Theta(\lambda) = \lim_{\tau \to \infty} \tau^{-1} \ln \prod_{n=-\infty}^{\infty} \frac{\det A^\lambda(\omega_n)}{\det A^0(\omega_n)} = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, \ln \left[\frac{\det A^\lambda(\omega)}{\det A^0(\omega)} \right]. \tag{B.9}$$

The determinant of a matrix of the form $A_{ij}^\lambda = c_i^{-1} \delta_{ij} + \lambda b$ satisfies the relation

$$\frac{\det A^\lambda}{\det A^0} = 1 + \lambda b \sum_i c_i; \tag{B.10}$$

we finally obtain

$$\phi_\Theta(\lambda) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, \ln \left[1 - \frac{2\alpha\lambda\omega \sum_i (T_i/\Theta)\nu_i(\omega)}{|D(\omega)|^2} \right]. \tag{B.11}$$

In general, *there is no choice of Θ such that $\phi_\Theta(\lambda)$ verifies the fluctuation theorem*, i.e. $\phi_\Theta(\lambda) \neq \phi_\Theta(1 - \lambda)$.

For the second definition, given by equation (80), the computation is identical to that of the previous section with the substitution $\Theta \to T(\omega)$, where $T(\omega)$ is given by equation (61). The result is then

$$\phi_{\mathrm{diss}}(\lambda) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, \ln \left[1 - \frac{2\alpha\lambda\omega \sum_i T_i \nu_i(\omega)}{T_{\mathrm{eff}}(\omega)|D(\omega)|^2} \right] = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \, \ln \left[1 - \frac{2\alpha\lambda\omega \sum_i \nu_i(\omega)}{|D(\omega)|^2} \right]. \tag{B.12}$$

Observing that

$$\begin{aligned} D(-\omega) &= \overline{D(\omega)} - 2\mathrm{i}\alpha, \\ |D(-\omega)|^2 &= |D(\omega)|^2 - 2\alpha\omega \sum_i \nu_i(\omega), \end{aligned} \tag{B.13}$$

it is now easy to show that $\phi_{\mathrm{diss}}(\lambda) = \phi_{\mathrm{diss}}(1 - \lambda)$.

Appendix C. Entropy production of the thermal baths

We will discuss here a different definition of the entropy production rate based on the power *extracted by the thermal bath* instead of that injected by the driving force. The two differ by a total derivative if there is only one bath, so their asymptotic distributions should be identical if boundary terms can be neglected. However, Van Zon and Cohen [40] showed in a particular case that this argument is not correct; see also [41].

If there are many baths equilibrated at different temperatures, the study of the entropy production extracted by each bath allows one to separate the different contributions to the total entropy production, weighting each one with the right temperature. We will first discuss the case of a single bath, and later we perform the computation for the general case. We will show that the entropy production rate defined in this way satisfies the fluctuation theorem as outlined in [42]; unfortunately, in our computation we neglect all boundary terms so we cannot check whether these terms modify the asymptotic distribution, as observed in [40, 41]. Anyway, the modification can be proven to be eventually relevant only for $|p| > 1$ [40, 41], so the results that we will discuss should hold *at least* for $|p| \leq 1$.

Appendix C.1. One bath

The entropy production rate of the bath is defined as

$$\sigma_t^{\text{bath}} = \beta \widetilde{W}_t = \beta[2\gamma \dot{z}_t \dot{\bar{z}}_t - 2\operatorname{Re} \dot{z}_t \bar{\rho}_t] = \sigma_t + \beta \frac{\mathrm{d}H}{\mathrm{d}t}, \tag{C.1}$$

where $\widetilde{W}_t$ has been defined in section 3 and is the power extracted from the system by the thermostat. Since σ_t and $\widetilde{\sigma}_t$ differ only by a total derivative (see equation (C.1)), for large τ we have

$$\mathcal{S}_\tau^{\text{bath}} = \mathcal{S}_\tau + \beta[H(\tau/2) - H(-\tau/2)] = \mathcal{S}_\tau + \beta \Delta H. \tag{C.2}$$

The first term in the rhs is $O(\tau)$ and has fluctuations $O(\sqrt{\tau})$, while the second term has zero average and its fluctuations are also $O(1)$. On these grounds, one usually neglects the second term and concludes that

$$\zeta_{\text{bath}}(p) = \zeta(p); \tag{C.3}$$

the two definitions of entropy production rate are equivalent and both distributions verify the fluctuation theorem. Van Zon and Cohen studied a model very similar to the one considered here, but where the distribution of $\mathcal{S}_\tau$ is Gaussian while the distribution of ΔH shows exponential tails. In such a case, the large fluctuations of $\mathcal{S}_\tau^{\text{bath}}$ are dominated by the distribution of ΔH even if $\langle (\mathcal{S}_\tau - \tau \sigma_+)^2 \rangle \gg \langle (\Delta H)^2 \rangle$; see [40, 41] and references therein. The modification of $\zeta_{\text{bath}}(p)$ occurs for $|p| > 1$; then, for $|p| \leq 1$, $\zeta_{\text{bath}}(p)$ still verifies the fluctuation relation.

Appendix C.2. Many baths

If the system is coupled to many equilibrated baths, in addition to the definitions of entropy production rate given by equations (85) and (80), a generalization of equation (C.2) is possible, given by the sum of the power dissipated by each bath divided by the corresponding temperature:

$$\sigma_t^{\text{baths}} = \sum_{i=1}^{N} \frac{\widetilde{W}_{it}}{T_i}. \tag{C.4}$$

This quantity takes into account heat exchanges between the baths, and its average value does not vanish at $\alpha = 0$, as we shall see in the following.

Let us compute $\phi_{\text{baths}}(\lambda)$. We have from equation (C.4)

$$\begin{aligned}
\mathcal{S}_\tau^{\text{baths}} &= \int_{-\tau/2}^{\tau/2} \mathrm{d}t \, \sigma_t^{\text{baths}} = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \operatorname{Re} \sum_i \frac{2}{T_i} \Big[\omega^2 |z_\omega|^2 g_i(\omega) + \mathrm{i}\omega z_\omega \bar{\rho}_{\mathrm{i}\omega} \Big] \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \left[\frac{\omega^2 |\sum_i \rho_{\mathrm{i}\omega}|^2 \sum_j \nu_j(\omega)/T_j}{|D(\omega)|^2} + \sum_{ij} \rho_{\mathrm{i}\omega} \bar{\rho}_{j\omega} \left(\frac{\mathrm{i}\omega}{D(\omega)T_j} - \frac{\mathrm{i}\omega}{\overline{D(\omega)}T_i} \right) \right].
\end{aligned} \tag{C.5}$$

If we define the functions
$$p(\omega) = i\omega\overline{D(\omega)},$$
$$F(\omega) = \omega^2 \sum_i \frac{\nu_i(\omega)}{T_i}, \tag{C.6}$$

we obtain

$$\langle \exp[-\lambda \mathcal{S}_\tau^{\text{baths}}] \rangle = \mathcal{N}^{-1} \int d\rho_{i\omega} \, \exp\left[-\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{ij} \rho_{i\omega} A_{ij}^\lambda(\omega) \bar\rho_{j\omega} \right], \tag{C.7}$$

where $A^\lambda(\omega)$ is an $N \times N$ matrix whose elements are given by

$$A_{ij}^\lambda(\omega) = \overline{A_{ji}^\lambda(\omega)} = \frac{\delta_{ij}}{T_i \nu_i(\omega)} + \frac{\lambda}{|D(\omega)|^2}\left[F(\omega) + \frac{p(\omega)}{T_j} + \frac{\overline{p(\omega)}}{T_i} \right]. \tag{C.8}$$

Then,

$$\phi_{\text{baths}}(\lambda) = \lim_{\tau \to \infty} \tau^{-1} \ln \prod_{n=-\infty}^{\infty} \frac{\det A^\lambda(\omega_n)}{\det A^0(\omega_n)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \ln\left[\frac{\det A^\lambda(\omega)}{\det A^0(\omega)} \right]. \tag{C.9}$$

The matrix A has the following form:

$$A \sim \begin{pmatrix} c_i^{-1} + \mu b_{ii} & \cdots & \mu b_{ij} \\ \vdots & \ddots & \vdots \\ \mu b_{ji} & \cdots & c_j^{-1} + \mu b_{jj} \end{pmatrix}, \tag{C.10}$$

where $\mu = \lambda/|D(\omega)|^2$, $c_i = T_i\nu_i(\omega)$ and $b_{ij} = F(\omega)+(p(\omega)/T_j)+(\overline{p(\omega)}/T_i)$. Its determinant is an order N polynomial in μ of the following form:

$$\frac{\det A^\lambda}{\det A^0} = 1 + \mu \sum_i c_i b_{ii} + \mu^2 \sum_{i<j} c_i c_j \begin{vmatrix} b_{ii} & b_{ij} \\ b_{ji} & b_{jj} \end{vmatrix} + \mu^3 \sum_{i<j<k} c_i c_j c_k \begin{vmatrix} b_{ii} & b_{ij} & b_{ik} \\ b_{ji} & b_{jj} & b_{jk} \\ b_{ki} & b_{kj} & b_{kk} \end{vmatrix} + \cdots. \tag{C.11}$$

Let us compute the coefficients explicitly. We will define T_{ij} by $T_{ij}^{-1} = T_i^{-1} - T_j^{-1}$. The coefficient of λ^2 is given by a sum of determinants of the form

$$\begin{vmatrix} F + \frac{p}{T_i} + \frac{\bar p}{T_i} & F + \frac{p}{T_i} + \frac{\bar p}{T_j} \\ F + \frac{p}{T_j} + \frac{\bar p}{T_i} & F + \frac{p}{T_j} + \frac{\bar p}{T_j} \end{vmatrix} = \begin{vmatrix} F + \frac{p}{T_i} + \frac{\bar p}{T_j} & \frac{\bar p}{T_{ji}} \\ F + \frac{p}{T_j} + \frac{\bar p}{T_i} & \frac{\bar p}{T_{ji}} \end{vmatrix} = \begin{vmatrix} \frac{p}{T_{ij}} & 0 \\ F + \frac{p}{T_j} + \frac{\bar p}{T_i} & \frac{\bar p}{T_{ji}} \end{vmatrix} = -\frac{|p|^2}{(T_{ij})^2}, \tag{C.12}$$

where we first subtracted the first column from the second column, and then subtracted the second row from the first row. We want now to show that all the coefficients of the higher powers of λ vanish. Consider for example the coefficient of λ^3. It has the form

$$\begin{vmatrix} F + \frac{p}{T_i} + \frac{\bar p}{T_i} & F + \frac{p}{T_i} + \frac{\bar p}{T_j} & F + \frac{p}{T_i} + \frac{\bar p}{T_k} \\ F + \frac{p}{T_j} + \frac{\bar p}{T_i} & F + \frac{p}{T_j} + \frac{\bar p}{T_j} & F + \frac{p}{T_j} + \frac{\bar p}{T_k} \\ F + \frac{p}{T_k} + \frac{\bar p}{T_i} & F + \frac{p}{T_k} + \frac{\bar p}{T_j} & F + \frac{p}{T_k} + \frac{\bar p}{T_k} \end{vmatrix}$$
$$= \begin{vmatrix} F + \frac{p}{T_i} + \frac{\bar p}{T_i} & \frac{\bar p}{T_{ji}} & \frac{\bar p}{T_{ki}} \\ F + \frac{p}{T_j} + \frac{\bar p}{T_i} & \frac{\bar p}{T_{ji}} & \frac{\bar p}{T_{ki}} \\ F + \frac{p}{T_k} + \frac{\bar p}{T_i} & \frac{\bar p}{T_{ji}} & \frac{\bar p}{T_{ki}} \end{vmatrix} = \begin{vmatrix} \frac{p}{T_{ik}} & 0 & 0 \\ \frac{p}{T_{jk}} & 0 & 0 \\ F + \frac{p}{T_k} + \frac{\bar p}{T_i} & \frac{\bar p}{T_{ji}} & \frac{\bar p}{T_{ki}} \end{vmatrix} = 0, \tag{C.13}$$

where we subtracted the first column from the second and third columns, and then subtracted the third row from the first and second rows. The same argument applies to all the other coefficients up to order N. Finally, we get

$$
\begin{aligned}
\frac{\det A^\lambda(\omega)}{\det A^0(\omega)} &= 1 + \frac{\lambda}{|D(\omega)|^2} \sum_i T_i \nu_i(\omega) \left[F(\omega) + \frac{p(\omega)}{T_i} + \frac{\overline{p(\omega)}}{T_i} \right] \\
&\quad - \frac{\lambda^2 |p(\omega)|^2}{|D(\omega)|^4} \sum_{i<j} \frac{T_i T_j \nu_i(\omega)\nu_j(\omega)}{(T_{ij})^2} = 1 - \frac{2\alpha\omega\lambda \sum_i \nu_i(\omega)}{|D(\omega)|^2} + \frac{\lambda(1-\lambda)}{|D(\omega)|^2} \\
&\quad \times \sum_{i<j} \frac{T_i T_j \nu_i(\omega)\nu_j(\omega)}{(T_{ij})^2},
\end{aligned}
\tag{C.14}
$$

and

$$
\phi_{\text{baths}}(\lambda) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \ln\left[1 - \frac{2\alpha\omega\lambda \sum_i \nu_i(\omega)}{|D(\omega)|^2} + \frac{\lambda(1-\lambda)}{|D(\omega)|^2} \sum_{i<j} \frac{T_i T_j \nu_i(\omega)\nu_j(\omega)}{(T_{ij})^2} \right]. \tag{C.15}
$$

The first term in the logarithm is proportional to α and is related to the power injected by the external force, while the second term accounts for heat exchanges between the baths and does not vanish at $\alpha = 0$. Finally, observing that

$$
\begin{aligned}
D(-\omega) &= \overline{D(\omega)} - 2\mathrm{i}\alpha, \\
|D(-\omega)|^2 &= |D(\omega)|^2 - 2\alpha\omega \sum_i \nu_i(\omega),
\end{aligned}
\tag{C.16}
$$

and using the trick that we already used above, it is easy to show that $\phi_{\text{baths}}(\lambda) = \phi_{\text{baths}}(1-\lambda)$. Thus $\zeta_{\text{baths}}(p)$ should verify the fluctuation relation *at least* for $|p| \leq 1$, if the contribution of boundary terms is not negligible. This result is of interest for the study of heat conduction and is similar to the one discussed in [42].

Appendix D. Correlation functions of the harmonic oscillator coupled to two baths

In the harmonic case, $\mathcal{V}(|z|^2) = k|z|^2$, the correlation function of a variable z_t, whose time evolution is given by equation (87), can be computed analytically [10]. In the frequency domain, equation (87) reads

$$
z_\omega = \frac{\rho_\omega^{\text{f}} + \rho_\omega^{\text{s}}}{\kappa - \mathrm{i}\omega\gamma_{\text{f}} - [\mathrm{i}\omega\gamma_{\text{s}}/(1 - \mathrm{i}\omega\tau_{\text{s}})]} \equiv \frac{\rho_\omega^{\text{f}} + \rho_\omega^{\text{s}}}{D(\omega)}, \tag{D.1}
$$

where $D(\omega) = \kappa - \mathrm{i}\omega\gamma_{\text{f}} - (\mathrm{i}\omega\gamma_{\text{s}}/(1 - \mathrm{i}\omega\tau_{\text{s}}))$. Recalling that $\langle \rho_\omega^{\text{f}} \rho_{\omega'}^{\text{f}} \rangle = 4\pi\gamma_{\text{f}} T_{\text{f}} \delta(\omega + \omega')$ and $\langle \rho_\omega^{\text{s}} \rho_{\omega'}^{\text{s}} \rangle = (4\pi\gamma_{\text{s}} T_{\text{s}}/(1 + \omega^2\tau_{\text{s}}^2))\delta(\omega + \omega')$, and defining $C(\omega)$ using $\langle z_\omega z_\omega' \rangle = 2\pi\delta(\omega + \omega')C(\omega)$, we get

$$
\begin{aligned}
R(\omega) &= \frac{\mathrm{d}z_\omega}{\mathrm{d}\rho_\omega^{\text{f}}} = \frac{1}{D(\omega)}, \\
C(\omega) &= \frac{2\gamma_{\text{f}} T_{\text{f}} + (2\gamma_{\text{s}} T_{\text{s}}/(1 + \omega^2\tau_{\text{s}}^2))}{|D(\omega)|^2}.
\end{aligned}
\tag{D.2}
$$

The function $(1 - \omega\tau_s)D(\omega)$ is a polynomial in ω and its zeros are given by $\omega = -i\gamma_\pm$ where

$$\gamma_\pm = \frac{1}{2\gamma_f\tau_s}\left[(\kappa\tau_s + \gamma_f + \gamma_s) \pm \sqrt{(\kappa\tau_s + \gamma_f + \gamma_s)^2 - 4\kappa\tau_s\gamma_f}\right], \tag{D.3}$$

and $\mathrm{Re}\,\gamma_\pm > 0$. The response function is then given by

$$R(t) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega t}\frac{1}{D(\omega)} = \frac{\theta(t)}{\gamma_f\tau_s}\left[\frac{1 - \gamma_+\tau_s}{\gamma_- - \gamma_+}e^{-\gamma_+ t} + \frac{1 - \gamma_-\tau_s}{\gamma_+ - \gamma_-}e^{-\gamma_- t}\right], \tag{D.4}$$

and the correlation function is given by

$$C(t) = \frac{1}{(\gamma_f\tau_s)^2}\left[\frac{\gamma_f T_f(1 - \gamma_+^2\tau_s^2) + \gamma_s T_s}{(\gamma_- - \gamma_+)(\bar{\gamma}_- + \gamma_+)\mathrm{Re}\,\gamma_+}e^{-\gamma_+ t} + \frac{\gamma_f T_f(1 - \gamma_-^2\tau_s^2) + \gamma_s T_s}{(\gamma_+ - \gamma_-)(\bar{\gamma}_+ + \gamma_-)\mathrm{Re}\,\gamma_-}e^{-\gamma_- t}\right]. \tag{D.5}$$

In the case $\alpha = 0$, and in the limit $\gamma_f \ll \gamma_s \ll k\tau_s$ where the timescales of the two baths are well separated, one obtains

$$\gamma_+ \sim \frac{k\tau_s + \gamma_s}{\gamma_f\tau_s},$$
$$\gamma_- \sim \frac{1}{\tau_s}\left(1 - \frac{\gamma_s}{\gamma_s + k\tau_s}\right), \tag{D.6}$$

and

$$C(t) = \frac{T_s\gamma_s\tau_s}{(k\tau_s + \gamma_s)^2}e^{-t/\tau_s} + \frac{T_f\tau_s}{k\tau_s + \gamma_f}\exp\left[\left(-\frac{k\tau_s + \gamma_s}{\gamma_f\tau_s}\right)t\right],$$
$$R(t) = \theta(t)\left\{\frac{\gamma_s}{(k\tau_s + \gamma_s)^2}e^{-t/\tau_s} + \frac{1}{\gamma_f}\exp\left[\left(-\frac{k\tau_s + \gamma_s}{\gamma_f\tau_s}\right)t\right]\right\}. \tag{D.7}$$

From the latter expressions it is easy to check that one has $R(t) \sim -\beta_f\theta(t)\dot{C}(t)$ for short times ($t \ll \tau_s$) and $R(t) \sim -\beta_s\theta(t)\dot{C}(t)$ for large times ($t \sim \tau_s$). The same behaviour is found in the limit of small dissipation (small α), as one can check by plotting the exact expressions for the functions $R(t)$ and $C(t)$.

Appendix E. The expression for σ^{diss} in the adiabatic approximation

We start from the expression (89) for σ_t^{diss} and from equation (91). Then (remember that $\int_{-\infty}^{t}\mathrm{d}s\,\delta(t - s) = \theta(0) = 1/2$ in our convention),

$$\int_{-\infty}^{t}\mathrm{d}t'\,T^{-1}(t - t')\dot{z}_t\bar{z}_{t'} = \frac{\dot{z}_t\bar{z}_t}{2T_f} + \frac{\gamma_s}{2\Omega T_f\gamma_f(\tau_s)^2}\left(1 - \frac{T_s}{T_f}\right)\int_{-\infty}^{t}\mathrm{d}t'\,e^{-\Omega(t-t')}\dot{z}_t\bar{z}_{t'}. \tag{E.1}$$

We make the substitution $z_t = H_t + w_t$ and neglect all the terms proportional to $H_t w_t$: indeed, such terms vanish when σ_t^{diss} is integrated over time intervals of the order of τ_s, as, on such timescales, $\langle w_t \rangle = 0$ while H is constant. The first term gives then

$$\frac{\dot{z}_t\bar{z}_t}{2T_f} = \frac{\dot{H}_t\bar{H}_t + \dot{w}_t\bar{w}_t}{2T_f}. \tag{E.2}$$

In the second term, as $\Omega \sim 1/\tau_s$, we make the approximation $\int_{-\infty}^{t}\mathrm{d}t'\,e^{-\Omega(t-t')}\bar{z}_{t'} \sim H_t/\Omega$ to obtain

$$\frac{\gamma_s}{2\Omega^2 T_f\gamma_f(\tau_s)^2}\left(1 - \frac{T_s}{T_f}\right)\dot{H}_t\bar{H}_t \sim \frac{1}{2T_s}\left(1 - \frac{T_s}{T_f}\right)\dot{H}_t\bar{H}_t. \tag{E.3}$$

The (imaginary part of the) sum of these two terms times 4α gives equation (108).

References

[1] Evans D J, Cohen E G D and Morriss G P, *Probability of second law violations in shearing steady states*, 1993 *Phys. Rev. Lett.* **71** 2401

[2] Gallavotti G and Cohen E G D, *Dynamical ensembles in nonequilibrium statistical mechanics*, 1995 *Phys. Rev. Lett.* **74** 2694

[3] Cugliandolo L F and Kurchan J, *Analytical solution of the off-equilibrium dynamics of a long range spin-glass model*, 1993 *Phys. Rev. Lett.* **71** 173

[4] Cugliandolo L F and Kurchan J, *On the out of equilibrium relaxation of the Sherrington–Kirkpatrick model*, 1994 *J. Phys. A: Math. Gen.* **27** 5749

[5] Gallavotti G and Cohen E G D, *Dynamical ensembles in stationary states*, 1995 *J. Stat. Phys.* **80** 931

[6] Kurchan J, *Fluctuation theorem for stochastic dynamics*, 1998 *J. Phys. A: Math. Gen.* **31** 3719

[7] Lebowitz J L and Spohn H, *A Gallavotti–Cohen-type symmetry in the large deviation functional for stochastic dynamics*, 1999 *J. Stat. Phys.* **95** 333

[8] Crisanti A and Ritort F, *Violation of the fluctuation-dissipation theorem in glassy systems: basic notions and the numerical evidence*, 2003 *J. Phys. A: Math. Gen.* **36** R181

[9] Cugliandolo L F, Kurchan J and Peliti L, *Energy flow, partial equilibration and effective temperatures in systems with slow dynamics*, 1997 *Phys. Rev.* E **55** 3898

[10] Cugliandolo L F and Kurchan J, *A scenario for the dynamics in the small entropy production limit, frontiers in magnetism*, 2000 *J. Phys. Soc. Japan Suppl.* A **69** 247 [cond-mat/9911086]

[11] Fielding S, Sollich P and Mayer P, *Fluctuation-dissipation relations and effective temperatures in simple non-mean field systems*, 2002 *J. Phys.: Condens. Matter* **14** 1683
Sollich P, *Fluctuation-dissipation relations in trap models*, 2003 *J. Phys. A: Math. Gen.* **36** 10807
Bertin E, Dauchot O and Droz M, *Non-equilibrium temperatures in steady-state systems with conserved energy*, 2004 *Preprint* cond-mat/0412071

[12] Sellitto M, *Fluctuations of entropy production in driven glasses*, 1998 *Preprint* cond-mat/9809186

[13] Semerjian G, Cugliandolo L F and Montanari A, *On the stochastic dynamics of disordered spin models*, 2004 *J. Stat. Phys.* **115** 493

[14] Crisanti A and Ritort F, *Intermittency of glassy relaxation and the emergence of a non-equilibrium spontaneous measure in the aging regime*, 2004 *Europhys. Lett.* **66** 253

[15] Sasa S, *A fluctuation theorem for phase turbulence of chemical oscillatory waves*, 2000 *Preprint* nlin.CD/0010026

[16] Bonetto F, Gallavotti G and Garrido P L, *Chaotic principle: an experimental test*, 1997 *Physica* D **105** 226

[17] Bonetto F, Chernov N I and Lebowitz J L, *(Global and local) fluctuations of phase-space contraction in deterministic stationary non-equilibrium*, 1998 *Chaos* **8** 823

[18] Gallavotti G, Rondoni L and Segre E, *Lyapunov spectra and nonequilibrium ensembles equivalence in 2D fluid mechanics*, 2004 *Physica* D **187** 338

[19] Zamponi F, Ruocco G and Angelani L, *Fluctuations of entropy production in the isokinetic ensemble*, 2004 *J. Stat. Phys.* **115** 1655

[20] Ciliberto S and Laroche C, *An experimental test of the Gallavotti–Cohen fluctuation theorem*, 1998 *J. Physique IV* **8** Pr6-215

[21] Goldburg W I, Goldschmidt Y Y and Kellay H, *Fluctuation and dissipation in liquid-crystal electroconvection*, 2001 *Phys. Rev. Lett.* **87** 245502

[22] Ciliberto S, Garnier N, Hernandez S, Lacpatia C, Pinton J-F and Ruiz Chavarria G, *Experimental test of the Gallavotti–Cohen fluctuation theorem in turbulent flows*, 2004 *Physica* A **340** 240

[23] Feitosa K and Menon N, *Fluidized granular medium as an instance of the fluctuation theorem*, 2004 *Phys. Rev. Lett.* **92** 164301

[24] Parisi G, *Off-equilibrium fluctuation dissipation relation in binary mixtures*, 1997 *Phys. Rev. Lett.* **79** 3660
Barrat J-L and Kob W, *Aging and the fluctuation dissipation ratio in a Lennard-Jones fluid*, 1999 *J. Phys. A: Math. Gen.* **11** A247
Barrat J-L and Kob W, *Fluctuation-dissipation ratio in an aging Lennard-Jones glass*, 1999 *Europhys. Lett.* **46** 637
Kob W and Barrat J-L, *Fluctuations, response and aging dynamics in a simple glass-forming liquid out of equilibrium*, 2000 *Eur. Phys. J.* B **13** 319
di Leonardo R, Angelani L, Parisi G and Ruocco G, *Off-equilibrium effective temperature in monoatomic Lennard-Jones glass*, 2000 *Phys. Rev. Lett.* **84** 6054

[25] Cugliandolo L F, Kurchan J, Le Doussal P and Peliti L, *Glassy behaviour in disordered systems with non relaxational dynamics*, 1997 *Phys. Rev. Lett.* **78** 350

[26] Le Doussal P, Cugliandolo L F and Peliti L, *Dynamics of particles and manifolds in a quenched random force field*, 1997 *Europhys. Lett.* **39** 111

  Corberi F, Gonnella G, Lippiello E and Zannetti M, *Correlation functions and fluctuation-dissipation relation in driven mixtures: an exactly solvable model*, 2003 *J. Phys. A: Math. Gen.* **36** 4729

[27] Berthier L, Barrat J L and Kurchan J, *A two-timescale, two-temperature scenario for nonlinear rheology*, 2000 *Phys. Rev. E* **61** 5464

[28] Barrat J L and Berthier L, *Fluctuation-dissipation relation in a sheared fluid*, 2000 *Phys. Rev. E* **63** 012503

[29] Berthier L and Barrat J L, *Nonequilibrium dynamics and fluctuation-dissipation relation in a sheared fluid*, 2002 *J. Chem. Phys.* **116** 6228

[30] Barrat A, Kurchan J, Loreto V and Sellitto M, *Edwards' measures for powders and glasses*, 2000 *Phys. Rev. Lett.* **85** 5034

  Kolton A, Exartier R, Domínguez D, Cugliandolo L F and Gronbech-Jensen N, *Effective temperature in driven vortex lattices with random pinning*, 2002 *Phys. Rev. Lett.* **89** 227001

  Hayashi K and Sasa S, *Extended Einstein relations with a complex effective temperature in a one dimensional driven lattice gas*, 2004 *Phys. Rev. E* **69** 066119

  Sastre F, Dornic I and Chaté H, *Bona fide thermodynamic temperature in nonequilibrium kinetic Ising models*, 2003 *Phys. Rev. Lett.* **91** 267205

  Arrachea L and Cugliandolo L F, *Study of a fluctuation-dissipation relation of a dissipative driven mesoscopic system*, 2005 *Europhys. Lett.* **70** 642

  Nakamura T, Otsuki M and Sasa S, *Anomalous time correlation in two-dimensional driven diffusive systems*, 2004 *Preprint* cond-mat/0410386

[31] Gallavotti G, *Extension of Onsager's reciprocity to large fields and the chaotic hypothesis*, 1996 *Phys. Rev. Lett.* **77** 4334

[32] Gallavotti G, *Chaotic hypothesis: Onsager reciprocity and fluctuation-dissipation theorem*, 1996 *J. Stat. Phys.* **84** 899

[33] Cugliandolo L F, *Dynamics of glassy systems*, 2002 *Slow Relaxation and Nonequilibrium Dynamics in Condensed Matter* (*Les Houches Session 77*) ed J-L Barrat *et al* (Berlin: Springer)

[34] Abou B and Gallet F, *Probing an nonequilibrium Einstein relation in an aging colloidal glass*, 2004 *Phys. Rev. Lett.* **93** 160603

[35] Pottier N and Mauger A, *Anomalous diffusion of a particle in an aging medium*, 2004 *Physica A* **332** 15

[36] Pottier N, *Out of equilibrium generalized Stokes–Einstein relation: determination of the effective temperature of an aging medium*, 2005 *Physica A* **345** 472

[37] Cugliandolo L F, *Heterogeneities and local fluctuations in glassy systems*, 2003 *SPIE* Vol. 5112, *Proc. 'Noise as a tool for studying materials', Symp. 'Noise and Fluctuations' (Santa Fe, June 2003)* ed M B Weissman, N E Israeloff and A S Kogan (Bellingham, WA: SPIE Optical Engineering Press) [cond-mat/0401506]

[38] Grier D G, *A revolution in optical manipulation*, 2003 *Nature* **424** 810

[39] Zamponi F, Ruocco G and Angelani L, *Generalized fluctuation relation and effective temperature in a driven fluid*, 2005 *Phys. Rev. E* **71** 020101(R)

[40] van Zon R and Cohen E G D, *Extended heat-fluctuation theorems for a system with deterministic and stochastic forces*, 2004 *Phys. Rev. E* **69** 056121

[41] Bonetto F, Gallavotti G, Giuliani A and Zamponi F, *Chaotic hypothesis, fluctuation theorem and singularities*, 2005 *Preprint* cond-mat/0507672

[42] Gallavotti G, *Entropy production in nonequilibrium stationary states: a point of view*, 2004 *Chaos* **14** 680

[43] Press W H, Flannery B P, Teukolsky S A and Vetterling W T, 1988 *Numerical Recipes in C: The Art of Scientific Computing* (Cambridge: Cambridge University Press)

[44] Wang G M, Sevick E M, Mittag E, Searles D J and Evans D J, *Experimental demonstration of violations of the second law of thermodynamics for small systems and short timescales*, 2002 *Phys. Rev. Lett.* **89** 050601

[45] Kurchan J, Peliti L and Sellitto M, *Aging in lattice-gas models with constrained dynamics*, 1997 *Europhys. Lett.* **39** 365

[46] Crisanti A, Picco M and Ritort F, 2005 in preparation

[47] Crooks G E, *Path ensemble averages in systems driven far from equilibrium*, 2000 *Phys. Rev. E* **61** 2361

  Crooks G E, *Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences*, 1999 *Phys. Rev. E* **60** 2721

[48] Franz S and Virasoro M A, *Quasi-equilibrium interpretation of ageing dynamics*, 2000 *J. Phys. A: Math. Gen.* **33** 891

  Corberi F, Lippiello E and Zannetti M, *Slow relaxation in the large-N model for phase ordering*, 2002 *Phys. Rev. E* **65** 046136