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Critical indices for the Yukawa$_2$ quantum field theory

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Abstract

The understanding of the Yukawa$_2$ quantum field theory is still incomplete if the fermionic mass is much smaller than the coupling. We analyze the Schwinger functions for small coupling uniformly in the mass and we find that the asymptotic behavior of the two-point Schwinger function is anomalous and described by two critical indices, related to the renormalization of the mass and of the wave function. The indices are explicitly computed by convergent series in the coupling. © 1997 Published by Elsevier Science B.V.

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1. Introduction

A rigorous understanding of the Yukawa$_2$ model, describing the interaction of a fermion field with a boson field in two dimensions, has been obtained until now only if the fermionic mass m is much larger than the coupling $\sqrt{\lambda}$, for small λ: the Schwinger functions can be written in this case as a convergent series in the coupling, see Refs. [1-3]. Integrating out the Bose field the Yukawa$_2$ becomes a purely fermionic theory with a *non*-local, current-current interaction so one can expect that for large distance scales its behavior is close to the Thirring model's one. What is known of the Thirring model is in some sense complementary to what is known in the Yukawa$_2$ model: the theory is completely understood only if the fermion has no mass, as in that case the model is solvable and its Schwinger functions can be explicitly exhibited, see Ref. [4]. In the massless case the theory shows an *anomalous behavior* which can be read from the asymptotic large distances decay of the fermionic two-point Schwinger

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function, given by a power law whose exponent is not 1, like in the free case, but $1+\eta$ with η dependent on the coupling. In the massive case there is some information on the spectrum [5-7], but the Schwinger functions are not known. It is believed that the theory, at least for small m, shows an *anomalous dimension* in the ψ and the $\bar\psi\psi$ fields. One can expect then that also the Yukawa$_2$ model shows a similar anomalous behavior which can be read from the asymptotic decay properties of the Schwinger function.

Aim of this paper is to provide a rigorous construction of the Yukawa$_2$ model *uniformly* in the fermion mass, in particular if $m \ll \sqrt{\lambda} \ll M$, where M is the boson mass, and to compute the asymptotic behavior of the two-point Schwinger function, so completing the theory of the weakly coupled Yukawa$_2$ model. In order to explain our results let us introduce an *Euclidean Bose field* ϕ_x and two *Euclidean Fermi fields* $\psi^\Lambda_x, \bar\psi^\Lambda_x$, $x=(x_0,\boldsymbol{x})$ and Λ is an *ultraviolet cut-off*, see below. The theory is *regularized* by assuming that Λ is finite and $|\boldsymbol{x}| \leqslant L, |x_0| \leqslant L$ with the fields having periodic boundary conditions. L is called the *infrared cut-off*.

The field ϕ_x has a propagator

$$G(x-y)=\int dk\,\frac{e^{ik(x-y)}}{k^2+M^2}\,. \tag{1}$$

Here and in the following $k \equiv (k_0,\boldsymbol{k}) = \frac{2\pi}{L}(n_0,\boldsymbol{n})$, $n_0, \boldsymbol{n}$ integers and $\int dk \equiv \frac{4\pi^2}{L^2}\sum_{n_0,\boldsymbol{n}}$. The two-component spinors $\psi^\Lambda_x, \bar\psi^\Lambda_x$ have a propagator

$$g^\Lambda(x-y)=\int dk\,e^{-k^2/\Lambda^2}\,e^{ik(x-y)/\hbar}\,\frac{\not{k}+mI}{k^2+m^2} \tag{2}$$

with $\not{k} = ik_0\gamma_0 + \boldsymbol{k}\gamma_1$

$$\gamma_0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \gamma_1=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

and I is the identity matrix. It is easy to check that the large distance asymptotic behavior of $\lim_{L,\Lambda\to\infty} g^\Lambda(x-y) = g(x-y)$ is discriminated by m; if $M^{-1} \leqslant |x-y| \leqslant m^{-1}$ than $g(x-y)$ decay with a power law with exponent 1 while if $|x-y| \geqslant m^{-1}$ then $|g(x-y)| \leqslant Cm\,e^{-\kappa m|x-y|}$ where C,κ are suitable constants.

The *regularized fermionic Schwinger functions* are given by the following functional integral:

$$\begin{aligned}&S^{\Lambda,L}(x_1,\dots,x_n,y_1,\dots,y_n)\\&\quad=\frac{\int P(d\psi^\Lambda)\int P(d\phi)\,e^{-V^\Lambda}\bar\psi^\Lambda_{x_1}\dots\bar\psi^\Lambda_{x_n}\psi^\Lambda_{y_1}\dots\psi^\Lambda_{y_n}}{\int P(d\psi^\Lambda)\int P(d\phi)\,e^{-V^\Lambda}}\,,\end{aligned} \tag{3}$$

where

$$V^\Lambda(\psi^\Lambda,\phi)=\sqrt{\lambda}\int dx\,\phi_x\bar\psi^\Lambda_x\psi^\Lambda_x+\frac{\lambda\alpha_\Lambda^2}{2}\int dx :\phi_x^2: -\sqrt{\lambda}H_\Lambda\int dx\,\phi_x$$

and :: is the normal order, $\sqrt{\lambda}\alpha_\Lambda$ is the *bosonic mass renormalization* and $\sqrt{\lambda}H_\Lambda$ induces a *fermionic mass renormalization*. The *bosonic integration* $\int P(d\phi)$ is a linear functional and it is defined over the monomials of Bose field by the commutative Wick rule with propagator Eq. (1); in the same way the *fermionic integration* $\int P(\psi^{\Lambda})$ is defined over the monomials of Fermi fields by the anticommutative Wick rule with propagator Eq. (2). Our main results are summarized by the following theorem:

Theorem 1. : There exist an $\varepsilon > 0$ and functions $\alpha_{\Lambda}, H_{\Lambda}$ such that, for $\frac{m}{M}, \frac{\sqrt{\lambda}}{M} \leqslant \varepsilon$ the limit $\lim_{L,\Lambda\to\infty} S^{L,\Lambda}(x,y) = S(x-y)$ exist and for $|x-y| > \frac{1}{m(0)}$ it is bounded by

$$|S(x-y)| \leqslant C\frac{m(0)}{Z(0)}\exp(-\kappa m(0)|x-y|)\,, \tag{4}$$

where C and κ are constants. For $\frac{1}{M} < |x-y| < \frac{1}{m(0)}$ we find, instead

$$|S(x-y)| \leqslant \frac{C}{M^{\eta_3}}|x-y|^{-1-\eta_3}\,, \tag{5}$$

where

$$m(0) = \frac{m^{1+\eta_2}}{M^{\eta_2}}\,, \qquad Z(0) = \left(\frac{m}{M}\right)^{-\eta_1} \tag{6}$$

and $\eta_1 = \beta_1\lambda^2 + O(\lambda^3)$, $\eta_2 = -\beta_2\lambda + O(\lambda^2)$, $\beta_1, \beta_2 > 0$ and $\eta_3 = \eta_1(1+\eta_2)^{-1}$.

The existence of all the Schwinger functions is an easy corollary of the theorem derivation. The above results show that the Yukawa$_2$ model, despite its extreme simplicity, have a very rich structure. If $\frac{\sqrt{\lambda}}{m} \ll 1$, according to Ref. [2], the asymptotic behavior of $S(x-y)$ is essentially the same of the free Schwinger function, Eq. (2), i.e. the presence of the interaction does not change the physical properties of the system. [1] The opposite situation $\frac{\sqrt{\lambda}}{m} \gg 1$ presents a different behavior. Like in the $\lambda = 0$ case one can distinguish two regions in the large distance behavior of $S(x-y)$, discriminated by a mass scale which is changed by the interaction from $O(m)$ to $O(m^{1+\eta_2})$. In the first region again $S(x-y)$ decays exponentially, but the decay rate is now $O(m^{1+\eta_2})$ instead of $O(m)$; moreover in the bound the constant multiplying the exponential, which in the free case is $O(1)$, is now $O(m^{\eta_1})$ and so vanishing as $m \to 0$. In the second region the behavior is again, like in the free case, given by a power law, but the exponent is not 1 but $1+\eta_3$. One can read then from the large distance behavior of $S(x-y)$ the anomalous dimension of the fields ψ and $\bar{\psi}\psi$. This is perhaps more clear if we note that, as a simple consequence of our analysis, we can write $S(x-y) = S_A(x-y) + \lambda S_B(x-y)$ with

$$S_A(x-y) = \int dk\,\frac{e^{ik(x-y)}}{Z(k)}\frac{\not{k} + m(k)I}{k^2 + m(k)^2} \tag{7}$$

[1] In this case in fact one can easily verify from the theorem derivation that $m(0) = m(1+O(\sqrt{\lambda}))$, $Z(0) = 1 + O(\sqrt{\lambda})$ and that the bounds Eq. (4) can be written replacing $m(0), Z(0)$ with $m, 1$ and simply changing the constants C, κ; moreover in Eq. (5) $|x-y|^{-\eta_3}$ has to be replaced by $1 + \sqrt{\lambda}A(x-y)$, with $A(x-y)$ bounded so that in this case $S(x-y)$ has the same infrared behavior that $g(x-y)$.

with $m(k), Z(k)$ two regular functions such that $|m(k) - m| = O(\frac{m\lambda}{M^2})$, $|Z(k) - 1| = O(\frac{\lambda}{M^2})$ for $|k| > M$ and $m(0), Z(0)$ are given by Eq. (6). Both S_A, S_B verifies the bounds Eq. (4), (5). In the massless case $m = 0$ one sees that the asymptotic behavior of $S(x - y)$ for the Yukawa$_2$ model is identical to the massless Thirring model one, what was expected. In the massive case a similar comparison is not possible as, as we said, the Schwinger function of the massive Thirring model is not known; anyway our results are in agreement with the known anomalous behavior of the mass spectrum of the Thirring model [5-7].

We refer to [3] for the ultraviolet part of the theory and we concentrate on the infrared one. The technique we use is the Wilsonian renormalization group in a way very close to the ones used in [8-10] for the Luttinger model; this is not strange as such a model can be considered equivalent to the massless Thirring model with an ultraviolet cut-off, see Ref. [11]. The main difference is that in the Yukawa$_2$ model there is, besides the wave function renormalization, also a mass term with an anomalous dimension which is lacking in the Lüttinger model. It is possible to control the renormalization group flow in the space of effective actions with mathematical rigor since the perturbative series for the new effective interaction in terms of the previous scale ones converges (exploiting cancellations due to the Fermi statistic) provided that the previous scale interaction is (in some sense) small. This was used to give a rigorous construction to the $d = 2$ Gross-Neveu model with large fermion mass, see Ref. [12], which is an *asymptotically free* theory. On the contrary in the Lüttinger model the effective interaction tends to a small non-trivial fixed point and the possibility of controlling the flow relies on the vanishing of the Beta function, proved in [8-10]. In the Yukawa$_2$ model the flow is controlled thanks to a *partial* vanishing of the Beta function, and the vanishing part coincides with the Lüttinger model Beta function.

Finally our results are of course valid for the massive Thirring model with an ultraviolet cut-off The solution of the ultraviolet problem of the massive Thirring model, which seems in the reach of actual methods, and our results would immediately imply the rigorous construction of this model and the determination of its Schwinger function asymptotic decay.

2. The ultraviolet problem

A power counting argument, see Ref. [3], shows that the counterterms in Eq. (3) have to be chosen as: $\alpha_\Lambda^2 = -\int dx \,\mathrm{Tr}\,(g^\Lambda(x) g^\Lambda(-x))$ and $H_\Lambda = g^{(\mathrm{uv})}(0)$; note that $\alpha_\Lambda, H_\Lambda$ are divergent as $\Lambda \to \infty$. We first consider the denominator of Eq. (3) and write it ("integrating the boson fields") as

$$\int P(d\psi^\Lambda) P(d\phi)\, e^{-V^\Lambda(\psi^\Lambda,\phi)} = \int P(d\psi^\Lambda)\, e^{-\bar{V}^\Lambda(\psi^\Lambda)}, \tag{8}$$

where

$$\bar{V}(\psi) = -\frac{\lambda}{2}\int dx\, dy\, G_{\alpha_\Lambda}(x-y)(\bar{\psi}_x\psi_x - H_\Lambda)(\bar{\psi}_y\psi_y - H_\Lambda) + \lambda K_\Lambda$$

with K_Λ a suitable constant and

$$G_{\alpha_\Lambda}(x-y) = \int dk\, \frac{e^{ik(x-y)}}{k^2 + M^2 + \lambda\alpha_\Lambda^2}\,. \tag{9}$$

It is convenient to decompose the propagator $g^\Lambda(x-y)$ separating the ultraviolet and infrared singularity

$$g^\Lambda(x-y) = g^{(\mathrm{uv})}(x-y) + g^{(\mathrm{ir})}(x-y)\,, \tag{10}$$

where

$$g^{(\mathrm{uv})}(x-y) = \int dk\,(e^{-k^2/\Lambda^2} - e^{-k^2})\, e^{ik(x-y)}\frac{\not{k} + mI}{k^2+m} = \int dk\, e^{ik(x-y)} g^{(\mathrm{uv})}(k)\,,$$

$$g^{(\mathrm{ir})}(x-y) = \int dk\, e^{-k^2}\, e^{ik(x-y)}\frac{\not{k} + mI}{k^2+m^2} = \int dk\, e^{ik(x-y)} g^{(\mathrm{ir})}(k)\,.$$

Eq. (10) allows us to represent ψ_x^Λ as the sum of two independent fields $\psi_x^{(\mathrm{uv})}$ and $\psi_x^{(\mathrm{ir})}$ with propagator $g^{(\mathrm{uv})}(x-y)$ and $g^{(\mathrm{ir})}(x-y)$. Let us denote by $\int\{\mathcal{D}\psi\, e^{-\int dk\,\bar{\psi}h(k)^{-1}\psi}\}$ the fermionic integration, acting on monomials of field $\psi, \bar{\psi}$ by the anticommutative Wick rule with propagator $\int dk\, e^{ik(x-y)}h(k)$. Using these definitions we can write

$$\int P(d\psi^\Lambda)\, e^{-\bar{V}^\Lambda(\psi^\Lambda)} = \int\{\mathcal{D}\psi^{(\mathrm{ir})}\, e^{-\int dk\,\bar{\psi}^{(\mathrm{ir})}(g^{(\mathrm{ir})}(k))^{-1}\psi^{(\mathrm{ir})}}\}\, e^{-V^{0,\Lambda}(\psi^{(\mathrm{ir})})}\,, \tag{11}$$

where, by definition,

$$e^{-V^{0,\Lambda}(\psi^{(\mathrm{ir})})} = \int\{\mathcal{D}\psi^{(\mathrm{uv})}\, e^{-\int dk\,\bar{\psi}^{(\mathrm{uv})}(g^{(\mathrm{uv})}(k))^{-1}\psi^{(\mathrm{uv})}}\}\, e^{-\bar{V}^\Lambda(\psi^{(\mathrm{ir})}+\psi^{(\mathrm{uv})})}\,, \tag{12}$$

where the meaning of Eqs. (11), (12) is that the r.h.s. is given by a perturbative series identical to the one obtained by the l.h.s. writing the exponential not in braces as a series and applying the anticommutative Wick rule to the terms obtained.

In [3] it is proved that, with the above choice of the counterterms, [2]

$$\begin{aligned}\lim_{\Lambda\to\infty} V^{0,\Lambda}(\psi) &\equiv V^0(\psi)\\ &= -\frac{\lambda}{2}\int dx\, dy\, G_{\alpha_0}(x-y)\bar{\psi}_x\psi_x\bar{\psi}_y\psi_y\\ &\quad + \sum_{m\geqslant 0, s_1,\ldots,s_m}\int dx_1\ldots dx_m\, W^m(x_1,\ldots,x_m)\psi_{x_1}^{s_1}\ldots\psi_{x_m}^{s_m}\,,\end{aligned} \tag{13}$$

where $G_{\alpha_0}(x-y)$ is given by Eq. (9), $\alpha_0^2 = O(\lambda)$, $\psi_x^s = \bar{\psi}_x$ or ψ_x depending if $s=1$ or -1 and $W^m(x_1,\ldots,x_m)$ are integrable over any $m-1$ variables. After the integration

[2] This statement was proved in [3] with $H_\Lambda = g^\Lambda(0)$ obtaining instead of Eq. (13) an analogous formula in which the first term is Wick ordered. Eq. (13) follows trivially from this results with our choice of H_Λ, which we find more natural as H_Λ admits a bound *uniform* in m, i.e. $H_\Lambda \leqslant C\Lambda$ with C independent on m.

of the ultraviolet field component, the problem is reduced to an essentially equivalent one but a purely infrared propagator and a new potential with terms of arbitrary degree in the fields.

3. The infrared integration

In order to perform the infrared integration, i.e. the study of Eq. (11), we find it convenient to consider the mass term as a perturbation of a massless field; the reason is that we look for results valid for small m uniformly. So we write

$$\begin{aligned}
&\int \{\mathcal{D}\psi^{(\leqslant 0)}\, e^{-\int dk\, C_0(k)\bar\psi^{(\leqslant 0)}(-\not{k}+m)\psi^{(\leqslant 0)}}\}\, e^{-V^0(\psi^{(\mathrm{ir})})} \\
&= \int \{\mathcal{D}\psi^{(\leqslant 0)}\, e^{-\int dk\, C_0(k)\bar\psi^{(\leqslant 0)}(-\not{k})\psi^{(\leqslant 0)}}\}\, e^{-m\int dk\, C_0(k)\bar\psi^{(\leqslant 0)}\psi^{(\leqslant 0)} - V^0(\psi^{(\leqslant 0)})}\,.
\end{aligned} \tag{14}$$

We decompose the Grassmanian integration $\{\mathcal{D}\psi^{(\leqslant 0)}\, e^{-\int dk\, C_0(k)\bar\psi^{(\leqslant 0)}(-\not{k})\psi^{(\leqslant 0)}}\}$ into a product of independent Grassmanian integrations. This can be done by setting $g^{(\mathrm{ir})}(x-y) = \sum_{h=-\infty}^{0} g^{(h)}(x-y)$ and by writing $\psi_x^{(\leqslant 0)} = \sum_{h=-\infty}^{0} \psi_x^{(h)}$, with $\psi_x^{(h)}$ being a family of Grassmanian fields with a vanishing "cross propagator" (i.e. an independent family of variables) and with a "self propagator":

$$g^{(h)}(x-y) = \int dk\, e^{ikx}(e^{-k^2\gamma^{-2h-2}} - e^{-k^2\gamma^{-2h}})\frac{\not{k}}{k^2} \equiv \int e^{ikx} f(k^2\gamma^{-2h})\frac{\not{k}}{k^2}\,, \tag{15}$$

where $\gamma > 1$.

Note that $g^{(h)}(x-y)$ has good scaling properties, i.e. $g^{(h)}(x-y) = \gamma^h g^0(\gamma^h(x-y))$ and $|g^{(h)}(x)| \leqslant A\gamma^h e^{-\kappa\gamma^h|x|}$, if A is a constant matrix and κ is a constant; moreover $|g^{(h)}(k)| \leqslant A\gamma^{-h} e^{-\kappa\gamma^{-h}|k|}$. We define $\psi_x^{(\leqslant h)} = \sum_{k=-\infty}^{h} \psi_x^{(k)}$ and we label the spin components of $\psi_x^{(h)}, \psi_x^{(\leqslant h)}$ by the index $\sigma = \pm 1$.

A naive definition of the *effective potential* is

$$e^{-V^h(\psi^{(\leqslant h)})} = \int \prod_{k=h+1}^{0} \{\mathcal{D}\psi^{(k)}\, e^{-\int dk\, f^{-1}(\gamma^{-2k}k^2)\bar\psi^{(k)}(-\not{k})\psi^{(k)}}\}\, e^{-\bar V^0(\psi^{(\leqslant 0)})}$$

with

$$\bar V^0(\psi^{(\leqslant 0)}) \equiv m\int dk\, C_0(k)\bar\psi^{(\leqslant 0)}\psi^{(\leqslant 0)} + V^0(\psi^{(\leqslant 0)})\,.$$

By a standard argument one can verify, see for instance Ref. [13], that $V^h(\psi)$ is given by a sum of terms of the form

$$\int dk_1 \ldots dk_n\, f^{n,h}(k_1,\ldots,k_n) \prod_{i=1}^{n} \psi_{k_i,\sigma_i}^{s_i}\, \delta\left(\sum_{i=1}^{n} \sigma_i k_i\right)$$

and $f^{n,h}$ are expressed by the sum of the values of suitable *Feynman diagrams*. A kth-order diagram contributing to $f^{n,h}$ can be obtained as usual from k graph elements, which represent the addends in Eq. (13), formed by vertices with m emerging oriented half-lines with indices k_i, $h_i > h$, symbolizing the fields $\psi_{k_i}^{h_i}$. One pairs the half-lines with consistent orientation and same indices and leaves n half-lines not paired. The value of the diagram is obtained by multiplying the propagators $g^{h_i}(k_i)$ associated with each paired line and the kernels in Eq. (13) associated to the vertices, and integrating this product over all the momenta k_i associated with paired half-lines.

As usual the maximal connected subgraphs formed by lines with scale $\geqslant h_v$ are called *clusters* with scale h_v and denoted by v; if h_v' is the maximal among the scales of the lines entering in or coming out the cluster, one can verify, by a power counting argument, see for instance Ref. [13], that the generic k-order diagram is bounded by $\frac{C^k}{k!}\varepsilon^k \prod_v \gamma^{-D_v(h_v-h_v')}$, where $h_v' < h_v$, C is a constant, $\varepsilon = \max(\lambda, m)$, the product runs over all the clusters v with scale h_v and $D_v > 0$ for each except the clusters with four external lines, for which $D_v = 0$, and the clusters with two external lines, for which $D_v = -1$. Then the graphs with clusters with four or two external lines do not admit a bound *uniform* in h and one has to set a different perturbation expansion. This is done, in the renormalization group framework, see Ref. [13], by introducing a *localization operator* in the following way. $\mathcal{L}V_m^h = 0$ if $m > 4$ while

$$\begin{aligned}
&\mathcal{L}\int \prod_{i=1}^{4} dk_i \sum_n f^{4,h}(k_1,k_2,k_3,k_4)\,\delta\left(\sum_{i=1}^{4}(s_i k_i)\right)\prod_{i=1}^{4}\psi_{k_i,\sigma_i}^{s_i,(\leqslant h)} \\
&\quad = f^{4,h}(0,0,0,0)\int \prod_{i=1}^{4} dk_i\,\delta(k_1+k_2-k_3-k_4)\prod_{i=1}^{4}\psi_{k_i,\sigma_i}^{s_i,(\leqslant h)},
\end{aligned} \tag{16}$$

where $s_1 = s_2 = -s_3 = -s_4 = +1$. Moreover, if $f^{2,h}(k_1,k_2) \equiv f^{2,h}(k)$:[3]

$$\begin{aligned}
&\mathcal{L}\int dk\, f^{2,h}(k)\psi_{k,\sigma_1}^{+,(\leqslant h)}\psi_{k,\sigma_2}^{-,(\leqslant h)} \\
&\quad = \int dk\,[f^{2,h}(0) + \boldsymbol{k}\tilde{\partial}_{\boldsymbol{k}} f^{2,h}(0) + k^0\tilde{\partial}_{k_0} f^{2,h}(0)]\psi_{k,\sigma_1}^{+,(\leqslant h)}\psi_{k,\sigma_2}^{-,(\leqslant h)}.
\end{aligned} \tag{17}$$

The *relevant part* of $V^h(\psi)$ is then

$$\begin{aligned}
&\mathcal{L}V^h(\psi^{\leqslant h}) = z_h F_\zeta^{\leqslant h} + \gamma^h s_h F_\sigma^{\leqslant h} - \lambda_h F_\lambda^{\leqslant h}, \qquad (18)\\
&F_\zeta^{\leqslant h} = -\int dk\, \not{k}\bar{\psi}_k^{(\leqslant h)}\psi_k^{(\leqslant h)}, \\
&F_\sigma^{\leqslant h} = \int dk\,\bar{\psi}_k^{(\leqslant h)}\psi_k^{(\leqslant h)}, \\
&F_\lambda^{\leqslant 0} = \int \prod_{i=1}^{4} dk_i \left(\bar{\psi}_{k_1}^{(\leqslant h)}\psi_{k_2}^{(\leqslant h)}\right)\left(\bar{\psi}_{k_3}^{(\leqslant h)}\psi_{k_4}^{(\leqslant h)}\right)\delta\left(\sum_i \varepsilon_i k_i\right). \qquad (19)
\end{aligned}$$

[3] By $\tilde{\partial}$ we mean the discrete derivatives.

Note that the a priori possible *non-covariant* terms produced by the localizations do vanish, in fact, by parity or symmetry arguments.

We then integrate the infrared scales in the following way, if $\mathcal{R} = 1 - \mathcal{L}$:

$$\begin{aligned}
&\int \{\mathcal{D}\psi^{(\leqslant 0)}\, e^{-\int dk\, C_0(k)\bar\psi^{(\leqslant 0)}(-\not k)\psi^{(\leqslant 0)}}\}\, e^{-\mathcal{L}V^0 - \mathcal{R}V^0} \\
&= \int \{\mathcal{D}\psi^{(\leqslant 1)}\, e^{-\int dk\, C_1(k)\bar\psi^{(\leqslant 1)}(-\not k)\psi^{(\leqslant 1)}}\}\, e^{-V^{-1}} \\
&= \int \{\mathcal{D}\psi^{(\leqslant 1)}\, e^{-\int dk\, C_1(k)\bar\psi^{(\leqslant 1)}(-\not k)\psi^{(\leqslant 1)}}\}\, e^{-\mathcal{L}V^{-1} - \mathcal{R}V^{-1}} \\
&= \dots ,
\end{aligned} \tag{20}$$

i.e. we perform each integration by writing the effective potential V^h as $\mathcal{L}V^h + \mathcal{R}V^h$. In this way V^h is a *renormalized series* of the running coupling constants $v_k = (s_k, z_k, g_k)$, $k > h$ and v_h obeys to a recursive relation, called *Beta function*, $v_h = \beta_h(v_{h+1}, \dots, v_0)$.

The effective potential is still given by a sum of Feynman graphs but each cluster v with two or four external lines with value f^{2,h_v} or f^{4,h_v} is replaced by $\mathcal{R}f^{2,h_v}$ and $\mathcal{R}f^{4,h_v}$ and the effect is that the kth-order diagram is now bounded by $\frac{C^k}{k!}\,\varepsilon^k \prod_v \gamma^{-(D_v + d_v)(h_v - h'_v)}$, where everything is again defined as above but $\varepsilon = (\max_{k>h} |v_k|)$ and d_v is 1 for the clusters with four external lines and 2 for the cluster with two external lines. In other words with respect to the $\mathcal{R} = \mathrm{Id}$ case there is in the estimates a factor $\gamma^{-(h_v - h'_v)}$ more for the clusters with four external lines and a factor more $\gamma^{-2(h_v - h'_v)}$ for the clusters with two external lines. It is standard to check that the renormalization produces these factors by writing explicitly the effect of the renormalization, for instance on a graph with four external lines:

$$\begin{aligned}
\mathcal{R}f^{4,h_v}(k_1,k_2,k_3,k_4) &= f^{4,h_v}(k_1,k_2,k_3,k_4) - f^{4,h_v}(0,0,0,0) \\
&= \int_0^1 dt \frac{\partial f^{4,h_v}(tk_1,tk_2,tk_3,tk_4)}{\partial t} .
\end{aligned}$$

The derivative acting on f^{4,h_v} has the effect that:

(i) Some propagator $g^i(k)$ in the cluster v with $i \geqslant h_v$ is derived and so it is bounded by $A\gamma^{-2i} e^{-\kappa\gamma^{-i}|k|}$, if A is a constant matrix, so there is an extra factor $\gamma^{-i} \leqslant \gamma^{-h_v}$ in the estimates with respect to the $\mathcal{R} = \mathrm{Id}$ case;

(ii) A factor k_i, $i = 1, \dots, 4$ is produced, which has the effect that in the estimates there is an extra (respect to the $\mathcal{R} = \mathrm{Id}$ case) factor $\gamma^{h'_v}$.

Each graph in the renormalized expansion for the effective potential is then bounded uniformly in h. Even more, estimating the fermionic expectations in the renormalized expansion for the effective potential by the *Grahm-Hadamard inequality*, one can prove the analyticity of the effective potential on scale h as a function of the running coupling constants v_k, $k > h$ in a small domain. This follows from Ref. [12] in which a renormalized expansion for the effective potential essentially identical to our one is discussed for the Gross–Neveu model.

However, we do not expect that $\max_{k>h}|v_k|$ *is small.* In fact there is, contrary to the massless case, a *relevant* coupling in the renormalization group sense, namely s_h. Moreover also the *marginal* couplings give problems as by a second-order computation

$$z_{h-1} = z_h + \beta_1 \lambda_h^2 , \qquad \lambda_{h-1} = \lambda_h$$

with $\beta_1 > 0$, so that the vanishing of the second-order beta function for λ_h has the effect that z_h grows without bound as $h \to -\infty$.

4. Anomalous scaling

To overcome the problem of the uncontrollable growth of the running coupling constants a natural approach is to renormalize the kinetic and the mass term, following the approach introduced in Ref. [14] and more recently used in a rigorous context in [8–10,13,15,16].

Setting $Z_0 = 1$, $V^0(Z_0\psi^{(\leqslant 0)}) = \tilde{V}^0(\psi^{(\leqslant 0)})$, $\sigma_0 = 0$, the anomalous scaling integration is defined *recursively* in the following way. Once the fields $\psi^0, \dots, \psi^{h+1}$ have been integrated we have to evaluate

$$\int \{\mathcal{D}\psi^{(\leqslant h)}\, e^{-\int dk\, C_h(k) Z_h \bar{\psi}^{(\leqslant h)}(-\not{k}+\sigma_h(k))\psi^{(\leqslant h)}}\}\, e^{-V^h(\sqrt{Z_h}\psi^{(\leqslant h)})} , \tag{21}$$

where $C_h^{-1}(k) = \sum_{k=-\infty}^{h} f(\gamma^{2k}k^2) = e^{-\gamma^{-2h}(k^2+k_0^2)}$ and as in the preceding section we write the effective potential, by the localization operator, Eqs. (16), (17), as the sum of its relevant part $\mathcal{L}V^h(\psi^{(\leqslant h)}) = s_h F_\sigma^{\leqslant h} + z_h F_z^{\leqslant h} - \lambda_h F_\lambda^{\leqslant h}$ and its irrelevant part. Note that, contrary to what was done in the preceding section, we write the running coupling constant multiplying $F_\sigma^{\leqslant h}$ simply as s_h and not as $\gamma^h s_h$.

Before attempting the integration of the fields $\psi^{(h)}$ we extract the terms quadratic in the fields out of $\mathcal{L}V^h$ and we take them into account by changing the propagator. We write expression (21) as

$$\int \{\mathcal{D}\psi^{(\leqslant h)}\, e^{-\int dk\, C_h(k) Z_{h-1}(k) \bar{\psi}^{(\leqslant h)}(-\not{k}+\sigma_{h-1}(k))\psi^{(\leqslant h)}}\}$$
$$\times e^{\lambda_h F_\lambda^{\leqslant h}(\sqrt{Z_h}\psi^{(\leqslant h)}) - \mathcal{R}V^h(\sqrt{Z_h}\psi^{(\leqslant h)})} , \tag{22}$$

where

$$Z_{h-1}(k) = Z_h + C_h^{-1}(k) Z_h z_h , \tag{23}$$
$$Z_{h-1}(k)\sigma_{h-1}(k) = Z_h\sigma_h(k) + Z_h C_h^{-1}(k) s_h$$

and $Z_{h-1} \equiv Z_{h-1}(0)$.

The equality between the expressions (21) and (22) is a compact form to state the equality between two series, absolutely convergent at any finite L. Now one can perform the integration with respect to $\psi^{(h)}$ writing Eq. (22) as

$$\int \{\mathcal{D}\psi^{(\leqslant h-1)} e^{-\int dk\, C_{h-1}(k) Z_{h-1}\bar{\psi}^{(\leqslant h-1)}(-\not{k}+\sigma_{h-1}(k))\psi^{(\leqslant h-1)}}\}$$
$$\times \int \{\mathcal{D}\psi^{(h)} e^{-\int dk\, \tilde{f}^{h}(k)^{-1} Z_{h-1}\bar{\psi}^{(h)}(-\not{k}+\sigma_{h-1}(k))\psi^{(h)}}\}$$
$$\times e^{l_h F_\lambda^{\leqslant h}(\sqrt{Z_{h-1}}\psi^{(\leqslant h)}) - \mathcal{R}V^h(\sqrt{Z_{h-1}}\psi^{(\leqslant h)})}, \tag{24}$$

where we have "rescaled" the fields in the effective potential, $l_h = (Z_h/Z_{h-1})^2\lambda_h$ and the second integration has a propagator given by

$$g^{(h)}(x-y) = \int dk\, e^{ik(x-y)} \frac{1}{Z_{h-1}} \frac{\not{k} + \sigma_{h-1} I}{k^2 + \sigma_{h-1}^2} \tilde{f}_h(k) \tag{25}$$

with

$$\tilde{f}_h(k) = Z_{h-1}\left[\frac{C_h^{-1}(k)}{Z_{h-1}(k)} - \frac{C_{h-1}^{-1}(k)}{Z_{h-1}}\right].$$

It is finally the time to perform the integration with respect to the field $\psi^{(h)}$ obtaining

$$\int \{\mathcal{D}\psi^{(\leqslant h-1)} e^{-\int dk\, C_{h-1}(k) Z_{h-1}\bar{\psi}^{(\leqslant h-1)}(-\not{k}+\sigma_{h-1}(k))\psi^{(\leqslant h-1)}}\} e^{-V^{h-1}(\sqrt{Z_{h-1}}\psi^{(\leqslant h-1)})}, \tag{26}$$

where $V^h(\sqrt{Z_{h-1}}\psi^{(\leqslant h-1)})$ is given by the sum of Feynman graphs in which the factors $\sigma_h(k), z_h$ appear only in the propagators Eq. (25) and their possible growth does not affect the convergence of the series. Note that expression (26) has the same form of expression (21) so that the procedure can be iterated.

It is convenient to write the propagator in the following way (for an analogous statement, and relative proof, see Ref. [16]):

$$\begin{aligned} g^{(h)}(x-y) &= g_l^{(h)}(x-y) + C_2^{(h)}(x-y), \\ g_l^{(h)}(x-y) &= \int dk\, \frac{e^{ikx}}{Z_h(k)} \frac{\tilde{f}(\gamma^{-2h}k^2)\not{k}}{k^2}, \end{aligned} \tag{27}$$

where, if κ, A are suitable constants,

$$|g_{l,\sigma,-\sigma}^{(h)}(x-y)| \leqslant A\frac{\gamma^h}{Z_h} e^{-\gamma^h\kappa|x-y|}$$

and

$$|C_{2,\sigma,\sigma'}^{(h)}(x-y)| \leqslant A\frac{\gamma^h}{Z_h}\left(\frac{\sigma_h}{\gamma^h}\right)^i e^{-\gamma^h\kappa|x-y|}$$

with $i=1$ if $\sigma=\sigma'$, $i=2$ if $\sigma=-\sigma'$ and $\sigma_h \equiv \sigma_h(0)$.

5. The flow of the renormalization group

The convergence of the series for the anomalous effective potential with $\sigma_k = 0$ for all k was proved in [10] (some technical simplifications are done in [15]). This means that, if

$$\max_{k\geqslant h} |g_k| \leqslant \bar{\varepsilon}_h\,, \qquad \max_{\substack{i=0,1,2\\ k\geqslant h}} \left|\frac{\sigma_k}{\gamma^k}\right|^i \leqslant \tilde{C}_h\,, \qquad \max_{k\geqslant h} \frac{Z_k}{Z_{k-1}} \leqslant e^{\beta_1\bar{\varepsilon}_h^2} \tag{28}$$

then the contribution of order k to the effective potential V^{h-1} is bounded by $C^k\tilde{C}_h^k\bar{\varepsilon}_h{}^k$, if C is a suitable constant; this follows by noting that the propagator, by Eq. (27), obeys to the same bound as in the $\sigma_k = 0$ case. The above bound does not depend on L and we can remove the *infrared cut-off*, i.e. we can take the limit $L \to -\infty$ on the effective potential. Of course it is possible to find a constant ε_h so that, if $\lambda \leqslant \varepsilon_h$ then $\bar{\varepsilon}_h \leqslant C^{-1}\tilde{C}_h^{-1}$ and the series for V^{h-1} are convergent, see Ref. [10].

However, one is interested in obtaining an ε_h *independent* on h but it is not obvious at all that this is possible and we have to study the beta function equation for the running coupling constants in order to understand their behavior.

The results of the preceding section imply, as in Refs. [10,15], that the Beta function equations are given by

$$\begin{aligned}
l_{h-1} &= l_h + G_\lambda^{1,h}(l_h;\ldots;l_0) + G_\lambda^{2,h}(l_h,\sigma_h;\ldots;l_0,\sigma_0)\,,\\
\sigma_{h-1} &= \sigma_h + G_\sigma^{1,h}(l_h,\sigma_h;\ldots;l_0,\sigma_0)\,,\\
\frac{Z_{h-1}}{Z_h} &= 1 + G_z^1(l_h;\ldots;l_0) + G_z^{2,h}(l_h,\sigma_h;\ldots;l_0,\sigma_0)\,,
\end{aligned} \tag{29}$$

where, if Eq. (28) holds, $|G_\lambda^{2,h}| \leqslant C_1\frac{\sigma_h}{\gamma^h}l_h$, $C_2\sigma_h l_h \leqslant G_\sigma^h \leqslant C_1\sigma_h l_h$, $C_1 l_h^2 \leqslant G_z \leqslant C_2 l_h^2$ and $|G_z^{2,h}| \leqslant C_1\frac{\sigma_h}{\gamma^h}l_h$, with $C_2 > C_1$ suitable positive constants.

In the equations for l_h we have divided, using Eq. (27), the beta function into two parts and the term $G_\lambda^{1,h}$ is given by a sum of integrals of products of the part $g_{l,\omega}^k(x-y)$ of the propagator while in the terms $G_\lambda^{2,h}$ at least a $C_2^k(x-y)$ term is involved. In the equation for σ_h we have used that at least a diagonal propagator has to be involved in the term contributing to the beta function. In the last equation again we have divided the beta function into two terms as it was done for l_h.

The Beta function is a *short memory* dynamical system in the sense that is a set of equations of the form $v_{h-1} = \beta_h(v_h, v_{h+1},\ldots,v_0)$, which behaves "essentially" as a system without memory $v_{h-1} = \beta_h(v_h,v_h,\ldots,v_h)$, (this is a consequence of the convergence of the Beta function as function of its arguments, see Ref. [10]). Even more, the convergence of Eq. (29) allows us to say that the lowest non-zero terms determine the evolution of the running coupling constants. All the terms contributing to the Beta function Eq. (29) are series starting from the second order except $G_\lambda^{1,h}$. This, however, generates a problem as one cannot exclude a priori that there is some non-vanishing term at some large order, and the evolution of the running coupling constants will depend critically on this unknown term. In other words, as the theory is not *asymptotically free* (contrary to the Gross–Neveu model studied in [12]), the convergence of the Beta function and the second-order Beta function are not enough to understand the system.

Fortunately it is possible to prove that

$$\lim_{h\to-\infty} G_\lambda^{1,h}(\mu;\dots;\mu)=0\,, \tag{30}$$

i.e. it is vanishing at *all* orders. In fact $G_\lambda^{1,h}$ coincides, up to terms vanishing as $h\to-\infty$ as $O(\gamma^h)$, with the correspondent quantity of the Lüttinger model,[4] and, by using the properties of the exact solution [17,18] one can prove that it is vanishing, see Refs. [8–10,15].[5]

By Eq. (30), the considerations after Eq. (28) and the "short memory properties" of the beta function (see Ref. [10]) it follows (see Ref. [16]) that if for $0\geqslant k\geqslant h$, Eq. (28) holds, then

$$\begin{aligned}&-c_1\lambda^2<|g_{h-1}-g_0|<c_2\lambda^2\,,\qquad -\lambda\beta_3c_3h\leqslant\log\left(\frac{|\sigma_{h-1}|}{|\sigma_0|}\right)\leqslant-\lambda\beta_3c_4h\\ &-\beta_1c_3\lambda^2h\leqslant\log(|Z_{h-1}|)\leqslant-\beta_1c_4\lambda^2h\end{aligned} \tag{31}$$

if c_i,β_i are suitable constant, i.e. the flow is essentially described by the second-order truncation of the beta function.

At the beginning of this section we said that the effective potential V^{h-1} is expressed by a convergent series if $\lambda\leqslant\varepsilon_h$, where ε_h is such that $\bar\varepsilon_h\leqslant C^{-1}\tilde C_h^{-1}$; $\bar\varepsilon_h$ and $\tilde C_h$ are defined in Eq. (28). Unfortunately, from Eq. (31) we see $\lim_{h\to-\infty}\tilde C_h=\infty$ and $\lim_{h\to-\infty}\varepsilon_h=0$ so that, given any h^*, we can prove the summability of the series for the effective potential V^h for $h>h^*$ in a domain dependent on h^*. We will see that it is possible to choose the scale h^* in such a way that the remaining scales can be integrated in a single step. A "natural" way to fix such scale is by requiring that $\tilde C_{h^*}\leqslant\tilde C$, if $\tilde C$ is a proper constant that we will fix below. One finds

$$\frac{\log_\gamma\tilde C^{-1}\sigma_0}{1+\lambda\beta_2c_3}\leqslant h^*\leqslant\frac{\ln\tilde C^{-1}\sigma_0+1}{1+\lambda\beta_2c_4}.$$

It remains to prove the summability of V^h for $h\leqslant h^*$. The study of the case $h\geqslant h^*$ was done in some sense by approximating the anomalous propagator Eq. (25) with the propagator of a massless theory by Eq. (27). This approximation is of course not reasonable for large $|h|$ corresponding to momenta negligible with respect to the mass term. But for the scales $h\leqslant h^*$ the bound

[4] This is not too surprising as the Lüttinger model, when the interaction range and the Fermi momentum shrink to 0, is equivalent to the massless Thirring model, if a suitable wave function normalization is done, see Ref. [11]; moreover the massless Thirring model and the massless Yukawa$_2$ have the same infrared beta function up to term $O(\gamma^h)$

[5] Of course it should be possible to prove Eq. (30) without using the exact solution but, possibly, exploiting some well-known symmetries of the Lagrangian, but they are not so immediate to exploit in this approach as they hold only when the regularizations are removed [19].

$$|g^{(\leqslant h^*)}_{\omega,\omega'}(x-y)| \leqslant A\frac{\gamma^{h^*}}{Z_{h^*}} e^{-\gamma^{h^*}\kappa|x-y|}$$

holds. Note in fact that $k^2+\sigma^2_{h^*}(k) \geqslant \sigma^2_{h^*}$, as follows from Eq. (23). In other words the propagator $g^{(\leqslant h^*)}_{\omega,\omega'}(x-y)$ obeys to the same bound of $g^{(h)}_{\omega,\omega'}(x-y)$ for $h \geqslant h^*$; our choice of h^* is done just to obtain this. This means that the integration of the scales between $-\infty$ and h^* is equivalent to the integration of a single scale in a massless theory, so that from [10,15] also the series expressing

$$\int P_{Z_{h^*}}(d\psi^{(\leqslant h^*)})\, e^{-V^{h^*}(\sqrt{Z_{h^*}})\psi^{(\leqslant h^*)})} \tag{32}$$

is convergent. The nth-order of the series in the running coupling constants for the effective potential for $h > h^*$ are bounded by $(\bar{\varepsilon}_h)^n \tilde{C}^n C_1^n$ while the term for $h \leqslant h^*$ are bounded by $(\bar{\varepsilon}_{h^*})^n \frac{C_2^n}{\tilde{C}^n}$ if Eq. (28) is verified for any $h = 0,\dots,h^*$, so that there is a non-ambiguous way to fix $\tilde{C}$ so that $\bar{\varepsilon}$ is the largest possible.

We have discussed the convergence of the denominator of Eq. (3) when the cut-offs Λ, L are removed; the numerator has a similar expansion, as one can easily check repeating the arguments in Section 4 for this case, see Ref. [10].

The results are

$$\frac{\int P(d\psi)\, e^{-V}\bar{\psi}_x\psi_y}{\int P(d\psi)\, e^{-V}} = S^{(\mathrm{uv})}(x-y) + \sum_{h=h^*}^{0} g^{(h)}(x-y) + \sum_{h=h^*}^{0} \bar{g}^{(h)}(x-y) \tag{33}$$

with

$$|S^{(\mathrm{uv})}(x-y)| \leqslant A\, e^{-\kappa|x-y|}, \qquad |\bar{g}^{(h)}_{\omega,\omega'}(x-y)| \leqslant \frac{A\gamma^h}{Z_h} e^{-\kappa\gamma^h|x-y|}$$

and we call $g^{(\leqslant h^*)}(x-y)$ simply $g^{(h^*)}(x-y)$.

From the above equation it is easy to state the results in the theorem. In fact the second term in Eq. (33) gives Eqs. (6) and (7), defining

$$\eta_1 = -\frac{\log(Z_{h^*})}{\log(m)}, \qquad 1+\eta_2 = \frac{\log(\sigma_{h^*})}{\log(m)}.$$

The bounds, Eqs. (4), (5), can be obtained by Eq. (33), as for $\gamma^{-h_x-1} < |x-y| \leqslant \gamma^{-h_x}$, $h_x > h^*$:

$$\sum_{h=0}^{h^*} |\bar{g}^{(h)}(x-y)| \leqslant A_1 \left[\sum_{h=h^*}^{h_x-1} \frac{\gamma^h}{Z_h} + \sum_{h=h_x}^{0} \frac{\gamma^h}{Z_h} e^{-\kappa\gamma^h|x-y|}\right] \leqslant A_2 \gamma^{h_x(1-\eta_3)}$$

while for $|x-y| \geqslant \sigma^{-1}_{h^*}$:

$$\sum_{h=0}^{h^*} |\bar{g}^{(h)}(x-y)| \leqslant A_1 \frac{\gamma^{h^*}}{Z_{h^*}} e^{-\kappa\gamma^{h^*}|x-y|} \left(1 + \sum_{h=1}^{h^*+1} \gamma^{(h-h^*)(1-\eta_3)} (\gamma^{h-h^*} - 1)^{-N}\right)$$

$$\leqslant A_2 \frac{\gamma^{h^*}}{Z_{h^*}} e^{-\kappa\gamma^{h^*}|x-y|} \tag{34}$$

where $N > 2$ and A_1, A_2 are positive constants.

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