MATH 2401, PRACTICE TEST 1

Let Q be the square $-\pi \le x \le \pi$, $-\pi \le y \le \pi$,

$$f(x,y) = \cos x + \cos y$$

and $\mathbf{r}(t) = \pi t\,\mathbf{i} + \pi t^2\,\mathbf{j}$.

1) Is Q an open or a closed set? Is it connected? Is it bounded?

   Q is closed, bounded and connected.

2) Compute the differential of $f(x,y)$

$$\nabla f(x,y) = -\sin x\,\mathbf{i} - \sin y\,\mathbf{j}$$

3) Compute the differential of $f(x,y)$ along the direction of $\mathbf{r}(t)$.

   $\mathbf{r}'(t) = \pi\,\mathbf{i} + 2\pi t\,\mathbf{j}$ so that

$$T(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + 2t\,\mathbf{j}}{\sqrt{1+4t^2}}$$

and

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -\frac{\sin \pi t + 2t \sin \pi t^2}{\sqrt{1+4t^2}}$$

4) Find the absolute maximum and the absolute minimum of $f(x,y)$ on Q.

   $\nabla f(x,y) = 0$ on Q on the 5 points $(0,0)$ and $(\pm\pi, \pm\pi)$.
   We have to check on the boundary. $f(x,\pi)$ has a maximim for $x = 0$ and a minimum for $x = \pm\pi$. The same is true for $f(x,-\pi)$ while $f(\pm\pi, y)$ has a maximum for $y = 0$ and a minimum for $y = \pm\pi$. It follows that $(\pm\pi, \pm\pi)$ are absolute minima where $f(\pm\pi, \pm\pi) = -2$. Moreover $f(0, \pm\pi) = f(\pm\pi, 0) = 1$ while $f(0,0) = 2$ so that $(0,0)$ is an absolute maxi,um with $f(0,0) = 2$.

6) Find the tangent plane to the surface $z = f(x,y)$ in the points $(0,0)$ and $(1,-1)$.

   The tangent plane is given by:

$$z - f(x_0, y_0) = \frac{\partial}{\partial x} f(x_0, y_0)(x - x_0) + \frac{\partial}{\partial y} f(x_0, y_0)(y - y_0)$$

so that the tangent plane at $(0,0)$ is

$$z = 2$$

and at $(1,-1)$ is

$$z = \sin 1(y - x + 2) + 2\cos 1$$

7) Let $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j}$. Find a point $\mathbf{c}$ such that $\nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) = f(\mathbf{b}) - f(\mathbf{a})$.

Note that $f(\mathbf{b}) - f(\mathbf{a}) = 0$ and $\mathbf{b} - \mathbf{a} = -2\,\mathbf{j}$ so that we looking for a point of the form $\mathbf{i} - (1 - 2t)\,\mathbf{j}$ such that $\frac{\partial}{\partial y} f(x, y) = -\sin y = 0$. We must have $y = 0$ so that $\mathbf{c} = \mathbf{i}$.

8) Find the absolute maximum and the absolute minimum of $f(x, y)$ on the portion of $\mathbf{r}(t)$ contained in Q.

Observe that $\mathbf{r}(0) = (0, 0)$ and $\mathbf{r}(\pm 1) = (\pm\pi, \pi)$ so that the absolute maximum is for $t = 0$ in $(0, 0)$ with $f(0, 0) = 2$ and the absolute minimum is for $t = \pm 1$ in $(\pm\pi, \pi)$ with $f(\pm\pi, \pi) = -2$.

9) Find the absolute maximum and the absolute minimum of $g(x, y) = (\cos x + \cos y)^2$ on Q.

$g(x, y)$ is the square of $f(x, y)$ so it is always positive and it is 0 for $\cos x = -cosy$, i.e. $x = y + \pi$. The maximum of $g(x, y)$ is the square of the absolute maximum or of the absolute minimum of $f(x, y)$, whichever is larger. So that the absolute minimum is 0 for $x - y = \pi$ and the absolute maximum is 4 for $(0, 0)$ and $(\pm\pi, \pm\pi)$.