MATH 3235 **Houdré**

TEST II

15th **November** 2018 **Name:** Solutions

Answer all questions; show all work; closed books, no calculators. THE HONOR CODE APPLIES

Problem	Points	Score
1	60	60
2	50	50
3	50	50
4	40	40
Total	200	200

Slightly changed problem 2.

1 (60 pts) Let X and Y be two random variables such that the vector (X, Y) is uniformly distributed over the region $R = \{(x, y) \in \mathbb{R}^2 : 0 < x < y < 1\}$.

(a) (15 pts) Find $\mathbb{P}(X + Y < 1)$

R 

Since (X, Y) is uniformly distributed on the region R, its density is given by $f(x,y) = 2$ on R and 0 elsewhere. So $\mathbb{P}(X+Y<1) = \int_0^{\frac{1}{2}}\int_0^{y} 2\,dx\,dy + \int_{\frac{1}{2}}^{1}\int_0^{1-y} 2\,dx\,dy$

$$= \int_0^{\frac{1}{2}} 2y\,dy + \int_{\frac{1}{2}}^{1} 2(1-y)\,dy = \frac{1}{4} + 2\cdot\frac{1}{2} - \left(1 - \frac{1}{4}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

(b) (10pts) Find f_X the marginal pdf of X and $\mathbb{E}X$.

$$f_X(x) = \int f(x,y)\,dy = \int_x^1 2\,dy = 2(1-x), \quad 0 < x < 1$$

(5 pts)

(c) (10 pts)

$$\mathbb{E}X = \int_0^1 x\,2(1-x)\,dx = \int_0^1 2x\,dx - \int_0^1 2x^2\,dx$$

$$= 1 - \frac{2}{3} = \frac{1}{3}.$$

(5 pts)

(d) (10 pts) Find $f_{X|Y}(x|y)$ the conditional pdf of X given that $Y = y$.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2}{\int_0^y f(x,y)\,dx} = \frac{2}{\int_0^y 2\,dx} = \frac{1}{y},$$

$$0 < y < 1$$

(e) (15 pts) Find $\mathbb{E}(X|Y)$.

$$\mathbb{E}(X|Y=y) = \int x\, f_{X|Y}(x|y)\,dx = \int_0^y \frac{x}{y}\,dx$$

$$= \frac{y^2}{2y} = \frac{y}{2},$$

$$0 < y < 1$$

So $\mathbb{E}(X|Y) = \dfrac{Y}{2}$.

2) X and Y are two IID standard exponential r.v. Let $W = 2X - Y$.

a) Find pdf of W (35pts)

b) $\mathbb{E}W^2$. (15pts)

Sol b): First note that we do not need to know a) to find b). Indeed,

$$\mathbb{E}W^2 = \mathbb{E}(2X-Y)^2 = \mathbb{E}(4X^2 - 4XY + Y^2)$$

$$= 4\,\mathbb{E}(X^2) - 4\,\mathbb{E}XY + \mathbb{E}Y^2$$

$$\underset{\text{II}}{=} 4\,\mathbb{E}X^2 - 4\,\mathbb{E}X\,\mathbb{E}Y + \mathbb{E}Y^2$$

Now $\mathbb{E}X = \mathbb{E}Y = 1$ and $\mathbb{E}X^2 = \int_0^\infty x^2 e^{-x}\,dx$

$$= \left[x^2(-e^{-x})\right]_0^\infty + 2\int_0^\infty x e^{-x}\,dx = 0 + 2\cdot 1 = 2.$$

So

$$\mathbb{E}W^2 = 4\cdot 2 - 4\cdot 1\cdot 1 + 2 = 6$$

Sol a) Since X and Y are $\perp\!\!\!\perp$, so are $2X$ and $-Y$. Then, $f_W = f_{2X} * f_{-Y}$.

Since $f_X(x) = e^{-x}, x>0$. So $f_{2X}(x) = \begin{cases} \frac{1}{2}e^{-x/2}, & x>0 \\ 0, & x\le 0. \end{cases}$

Similarly $f_{-Y}(y) = e^{y}, y<0$, and 0 elsewhere

$$f_W(w) = \int_{-\infty}^{+\infty} f_{2X}(w-t)\, f_{-Y}(t)\,dt. \text{ Now,}$$

the integrand is 0 if $t \geq 0$ and $w \leq t$

So if $w \geq 0$, $f_W(w) = \int_{-\infty}^{0} \frac{1}{2} e^{-\frac{(w-t)}{2}} e^{t}\,dt$

$= \frac{1}{3} e^{-w/2}$. If $w<0$, $t<w$ and

$t<w \iff t\leq w$. So

$$\int_{-\infty}^{w} \frac{1}{2} e^{-\frac{1}{2}(w-t)} e^{t}\,dt = \frac{1}{3}e^{w}, \quad w<0$$

3. (50pts) Let X and Y have joint density function given by

$$f(x,y) = \frac{1}{y} e^{-y-\frac{x}{y}}, \quad \text{if} \quad x > 0, y > 0,$$

(and, as usual, it is zero elsewhere). Find the covariance of X and Y. Are X and Y positively correlated?

$$\mathbb{E}Y = \iint y f(x,y)\,dx\,dy = \int_0^{+\infty}\int_0^{\infty} e^{-y} e^{-\frac{x}{y}}\,dx\,dy = \int_0^{\infty} e^{-y}\int_0^{\infty} e^{-\frac{x}{y}}\,dx\,dy$$

$$= \int_0^{\infty} e^{-y}\left[-y e^{-x/y}\right]_0^{\infty} dy = \int_0^{\infty} y e^{-y}\,dy = 1 \qquad (10\text{pts})$$

$$\mathbb{E}X = \int_0^{\infty}\int_0^{\infty} \frac{x}{y} e^{-y} e^{-\frac{x}{y}}\,dx\,dy = \int_0^{\infty} \frac{e^{-y}}{y} \int_0^{\infty} x e^{-\frac{x}{y}}\,dx\,dy$$

$$= \int_0^{\infty} \frac{e^{-y}}{y}\left\{\left[x(-y e^{-x/y})\right]_0^{\infty} + \int_0^{\infty} y e^{-x/y}\,dx\right\} dy$$

$$= \int_0^{\infty} e^{-y}\left[-y e^{-x/y}\right]_0^{\infty} dy = \int_0^{\infty} y e^{-y}\,dy = 1 \qquad (10\text{pts})$$

$$\mathbb{E}XY = \int_0^{\infty}\int_0^{\infty} x e^{-y} e^{-\frac{x}{y}}\,dx\,dy = \int_0^{\infty} e^{-y}\int_0^{\infty} x e^{-\frac{x}{y}}\,dx\,dy$$

$$= \int_0^{\infty} e^{-y}\left\{\left[x(-y e^{-x/y})\right]_0^{\infty} + \int_0^{\infty} y e^{-x/y}\,dx\right\} dy$$

$$= \int_0^{\infty} e^{-y} y \int_0^{\infty} e^{-x/y}\,dx\,dy = \int_0^{\infty} y^2 e^{-y}\,dy = 2. \qquad (20\text{pts})$$

So $\mathrm{Cov}(X,Y) = \mathbb{E}XY - \mathbb{E}X\,\mathbb{E}Y = 2 - 1\cdot 1 = 1$ (5pts). Since $\mathrm{Cov}(X,Y) > 0$, X and Y are positively correlated (5pts)

(4) (40 pts). Recall that a standard Cauchy r.v. has pdf $f(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$ and c.f. $\underline{\varphi}(t) = e^{-|t|}$, $t \in \mathbb{R}$.

Let $X_1, \dots, X_{43}$ be iid standard Cauchy and let $S_{43} = \frac{X_1 + \dots + X_{43}}{43}$.

Then,

$$\underline{\varphi}_{S_{37}}(t) = \mathbb{E}\left(e^{it\left(\frac{X_1 + \dots + X_{43}}{43}\right)} \right)$$

$$\underline{\perp\!\!\!\perp} \quad = \mathbb{E}\, e^{\frac{itX_1}{43}} \cdots \mathbb{E}\, e^{\frac{itX_{43}}{43}}$$

identically distributed $\quad = \left(\mathbb{E}\, e^{\frac{itX_1}{43}} \right)^{43}$

$$= \left(e^{-\frac{|t|}{43}}\right)^{43}$$

Cauchy

$$= e^{-|t|}$$!

Hence by the uniqueness of the c.f.,

S_{43} is a standard Cauchy r.v. Hence its pdf is

$$f_{S_{43}}(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}.$$

Note that this exercise is problem 18 on ~~H~~HW7