No books or notes allowed. No laptop or wireless devices allowed. Show all your work for full credit. **Write clearly and legibly**.

Name (print): ______________________________

| Question: | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|
| Points: | 15 | 25 | 15 | 30 | 15 | 100 |
| Score: | | | | | | |

Question 1 ........................................................................ *15 point*

(a) (15 points) Let $X$ be a geometric r.v with parameters $p$. Compute the probability generating function $G_X(s)$ of $X$.

**Solution:** We have

$$G_X(s) = \sum_{x=1}^{\infty} p(1-p)^{x-1} s^x = \frac{ps}{1-(1-p)s}.$$

Question 2 .................................................................... *25 point*

Let $A, B$ and $C$ be three events. You know that

$$B \perp C, \qquad A \perp (B \cup C), \qquad A \perp (B \cap C), \qquad \text{and} \qquad A \perp (B \cap C^c)$$

where $B \perp C$ means that $B$ and $C$ are independent.

(a) (10 points) Show that

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$$

**Solution:** Since $A \perp B \cap C$ we have

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B \cap C)$$

and since $B \perp C$ we get

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$$

(b) (15 points) Show that

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

You can use that $B = (B \cap C) \cup (B \cap C^c)$.

**Solution:** We have

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \cap ((B \cap C) \cup (B \cap C^c))) = \mathbb{P}((A \cap B \cap C) \cup (A \cap B \cap C^c))$$

Since $(A \cap B \cap C) \cap (A \cap B \cap C^c)) = \emptyset$ we get

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \cap B \cap C) + \mathbb{P}(A \cap B \cap C^c)$$

using point (a) we get

$$\begin{aligned}\mathbb{P}(A \cap B) =& \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) + \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C^c) = \\ & \mathbb{P}(A)\mathbb{P}(B)(\mathbb{P}(C) + \mathbb{P}(C^c)) = \mathbb{P}(A)\mathbb{P}(B)\end{aligned}$$

Question 3 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . *15 point*

You and a friend of yours both toss a fair coin twice. Let $X_1$ be the number of heads you get and $X_2$ be the number of heads your friend gets. Compute the p.m.f. of $Y = X_1+X_2$ and $Z = X_1 - X_2$.

**Solution:** We clearly have

$$p_{X_1}(0) = p_{X_1}(2) = \frac{1}{4} \qquad p_{X_1}(1) = \frac{1}{2}$$

and similarly for $p_{x_2}$.

Thus

$$\begin{aligned}
&p_Y(0) = p_{X_1}(0)p_{X_2}(0) = \frac{1}{16}, \quad p_Y(1) = 2p_{X_1}(0)p_{X_2}(1) = \frac{1}{4},\\
&p_Y(2) = p_{X_1}(1)p_{X_2}(1) + 2p_{X_1}(0)p_{X_2}(2) = \frac{3}{8}.\\
&p_Y(3) = 2p_{X_1}(2)p_{X_2}(1) = \frac{1}{4}, \quad p_Y(4) = 2p_{X_1}(2)p_{X_2}(1) = \frac{1}{16}
\end{aligned}$$

On the other call $W = 2 - X_2$. Clearly $W$ is the number of tails your friend gets and has the same p.m.f of $X_2$. Moreover $Z = X_1 + W - 2$ so that

$$p_Z(z) = p_Y(z+2).$$

Question 4 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . *30 point*

You roll a fair dye till you get a 6. Let $X$ be the total number of roll and $Y$ be the number of 1 you see in this $X$ rolls.

(a) (15 points) Find $\mathbb{P}(Y = y|X = x)$ and $\mathbb{E}(Y|X = x)$.

**Solution:** Once $X$ is given all the previous $x-1$ outcomes must be in $\{1, 2, 3, 4, 5\}$. Each of these outcomes is equally probable and they are still independent. Thus

$$\mathbb{P}(Y = y|X = x) = \binom{x-1}{y}\left(\frac{1}{5}\right)^{y}\left(\frac{4}{5}\right)^{x-1-y}$$

and thus

$$\mathbb{E}(Y|X = x) = \frac{x-1}{5}.$$

(b) (15 points) Find $\mathbb{E}(Y)$.

**Solution:** We have

$$\mathbb{E}(Y) = \sum_{x=1}^{\infty}\mathbb{E}(Y|X = x)\mathbb{P}(X = x) = \sum_{x=1}^{\infty}\frac{x-1}{5}\mathbb{P}(X = x) =$$
$$\frac{1}{5}\left(\mathbb{E}(X) - 1\right) = \frac{5}{5} = 1$$

where we have used that $X$ is a geometric r.v. with $p = 1/6$.

Question 5 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . *15 point*

A large hall in your building require 100 light bulbs. You receive a shipment of 100 light bulbs and put them on. Each bulb breaks down in the first month of operation with probability $p = 0.1$. If it survives the first month, it breaks down in the second month of operation with probability $p = 0.15$. Suppose that after two months the number of working bulbs is 80. Obtain the p.m.f. of the number $X$ of bulbs that were working after one month.

**Solution:** Clearly the 80 bulbs that are working after two months were working also after one month. For the remaining 20, the fact that they were working after one month are independent events. Let $A$ be the event $\{\textit{bulb is broken after two mouths}\}$ and $B$ the event $\{\textit{bulb was broken after one mouth}\}$. We need

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B)}{\mathbb{P}(A)}$$

but

$$\mathbb{P}(B) = 0.1 \qquad \mathbb{P}(A) = \mathbb{P}(B) + 0.15\mathbb{P}(B^c) = 0.1 + 0.9 \cdot 0.15 = 0.23$$

so that

$$\mathbb{P}(B|A) = \frac{0.1}{0.1 + 0.9 \cdot 0.15} = 0.43$$

and $X = 80 + Y$ where $Y$ in binomial with parameters 20 and 0.43.

**Useful Formulas**

- If $A$ and $B$ are events then the probability of $A$ given $B$ is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

  and $A$ and $B$ are independent if $\mathbb{P}(A|B) = \mathbb{P}(A)$ or $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

- If $X$ is a binomial r.v. with parameters $N$ and $p$ then

$$bin(x; N, p) = \mathbb{P}(X = x) = \binom{N}{x} p^x (1-p)^{N-x}.$$

  Moreover $\mathbb{E}(X) = Np$ and $\text{var}(X) = Np(1-p)$.

- If $X$ is a geometric r.v. with parameter $p$ then

$$p(x) = \mathbb{P}(X = x) = (1-p)^{x-1} p.$$

  Moreover $\mathbb{E}(X) = 1/p$.

- The probability generating function $G_X(s)$ of a r.v. $X$ is defined by

$$G_X(s) = \mathbb{E}\left(s^X\right).$$