No books or notes allowed. No laptop or wireless devices allowed. Show all your work for full credit. **Write clearly and legibly**.

Name (print): ______________________________

Question:	1	2	3	4	5	Total
Points:	40	20	30	20	0	110
Score:						

Question:	1	2	3	4	5	Total
Bonus Points:	0	0	0	0	20	20
Score:						

Question 1 . . . . 40 point

Let X be a continuous r.v. with p.d.f. given by

$$f(x) = \begin{cases} p + 2(1-p)x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where p is a parameter.

(a) (10 points) for which value of p is f a valid p.d.f. (**Hint**: remember that there are 2 conditions you should check.)

> **Solution:** We have
>
> $$\int_0^1 f(x)dx = 1$$
>
> for every p so that we just need to check that $f(x) > 0$. Since f is linear in x it is enough that $f(0) > 0$ and $f(1) > 0$. Thus we get
>
> $$0 \le p \le 2\,.$$

(b) (15 points) Compute the expected value $\mathbb{E}(X)$ and the variance $V(X)$ of X.

> **Solution:** We have
>
> $$\mathbb{E}(X) = \int_0^1 xf(x)dx = p\int_0^1 xdx + 2(1-p)\int_0^1 x^2dx = \frac{p}{2} + \frac{2}{3}(1-p) = \frac{2}{3} - \frac{p}{6}$$
>
> and
>
> $$\mathbb{E}(X^2) = \int_0^1 x^2f(x)dx = p\int_0^1 x^2dx + 2(1-p)\int_0^1 x^3dx = \frac{p}{3} + \frac{1}{2}(1-p) = \frac{1}{2} - \frac{p}{6}$$
>
> so that
>
> $$V(x) = \frac{1}{36}\left(p\,(2-p) + 2\right)$$

(c) (15 points) Show that

$$\mathbb{E}\left(\left(X-\frac{2}{3}\right)^2\right) \geq \left(\frac{p}{6}\right)^2$$

(**Hint**: use Jensen inequality.)

Solution: Use Jensen inequality to get

$$\mathbb{E}\left(\left(X-\frac{2}{3}\right)^2\right) \geq \left(\mathbb{E}(X)-\frac{2}{3}\right)^2 = \left(\frac{1}{2}-\frac{p}{6}-\frac{1}{2}\right)^2 .$$

Question 2 . . . . *20 point*

Let N_k, $k = 1, 2, 3, \ldots$, be an infinite sequence of geometric random variable with parameter $p_k = \frac{\lambda}{k}$, that is

$$\mathbb{P}(N_k = n) = (1 - p_k)^{n-1} p_k \qquad \text{for } n \geq 1\,.$$

and $\mathbb{P}(N_k = n) = 0$ for $n < 1$. Moreover let Y be an exponential r.v. with parameter λ, that is

$$f_Y(y) = \lambda e^{-\lambda y} \qquad \text{for } y \geq 0$$

and $f_Y(y) = 0$ for $y < 0$.

Show that $Z_k = N_k/k$ converge in distribution to Y as $k \to \infty$. (**Hint**: compute the c.d.f. of Z_k, that is $F_k(x) = \mathbb{P}(Z_k \leq x)$ for every real number x.)

Solution: Observe that we have

$$\mathbb{P}(Z_k \leq x) = \mathbb{P}(N_k \leq kx) = \sum_{i=1}^{\lfloor kx \rfloor} (1 - p_k)^{i-1} p_k = 1 - (1 - p_k)^{\lfloor kx \rfloor - 1} = \left(1 - \frac{\lambda}{k}\right)^{\lfloor kx \rfloor - 1}$$

while

$$\mathbb{P}(Y \leq y) = 1 - e^{-\lambda y}.$$

Observe that

$$\frac{\lfloor kx \rfloor - 1}{k} \to x$$

as $k \to \infty$ so that

$$\left(1 - \frac{\lambda}{k}\right)^{\lfloor kx \rfloor - 1} = \left(\left(1 - \frac{\lambda}{k}\right)^{k}\right)^{\frac{\lfloor kx \rfloor - 1}{k}} \to_{k \to \infty} e^{-\lambda x}$$

for every x.

Question 3 .... *30 point*

A student is attempting a multiple choices exam. For each question there are 4 possible answers. He has a probability of 0.75 of knowing the correct answer. If he does not know the answer he chooses one answer uniformly and randomly. All questions and answers are independent.

To get a B he need to answer correctly 85% of the questions while to get an A he needs to answer correctly 95% of the questions.

(a) (15 points) If the test contains 40 questions, use a normal approximation (CLT) and the table provided to compute the probability p_B that the student will get at least a B and the probability p_A that the student will get a A.

Solution: Let p be the probability that the student give a correct answer. We have

$$p = 0.75 + 0.25 \cdot 0.25 = 0.8125\,.$$

Let X_i be 1 if he answer correctly to the i-th question and 0 otherwise. Thus $\mathbb{E}(X_i) = 0.8125$ and $V(X_i) = 0.1523$. We get

$$\begin{aligned} p_B =&\mathbb{P}\left(\sum_{i=1}^{40} X_i > 0.85 \cdot 40\right) = \\ =&\mathbb{P}\left(\frac{\sum_{i=1}^{40} X_i - 0.8125 \cdot 40}{0.390\sqrt{40}} > \frac{(0.85 - 0.8125) \cdot 40}{0.390\sqrt{40}}\right) = \\ =&1 - \Phi(0.60) = 0.274 \end{aligned}$$

while

$$\begin{aligned} p_B =&\mathbb{P}\left(\sum_{i=1}^{40} X_i > 0.95 \cdot 40\right) = \\ =&\mathbb{P}\left(\frac{\sum_{i=1}^{40} X_i - 0.8125 \cdot 40}{0.390\sqrt{40}} > \frac{(0.95 - 0.8125) \cdot 40}{0.390\sqrt{40}}\right) = \\ =&1 - \Phi(2.23) = 0.013\,. \end{aligned}$$

(b) (15 points) Let p_B be the probability that a student that knows 75% of the answers will get a B or more. If the teacher wants p_B to be less than 0.025, how many question should there be on the exam.

Solution: He wants to find N such that

$$\mathbb{P}\left(\sum_{i=1}^{N} X_i > 0.85 \cdot N\right) \leq 0.025$$

This means

$$\mathbb{P}\left(\frac{\sum_{i=1}^{N} X_i - 0.8125 \cdot N}{0.390\sqrt{N}} > \frac{(0.85 - 0.8125) \cdot \sqrt{N}}{0.390}\right) = 1 - \Phi(0.096 \cdot \sqrt{N}) \leq 0.025$$

From the table we

$$\Phi(1.96) = 0.975$$

so that he needs

$$N > \left(\frac{1.96}{0.096}\right)^2 = 416$$

questions.

Question 4 .... *20 point*

Let Z be a standard normal random variable. Find the n-th moment $m_n = \mathbb{E}(Z^n)$ of Z, for every n. (**Hint**: you can use integration by part to relate m_n with m_{n-2} and then use induction. Alternatively you can use the Taylor expansion around 0 of the m.g.f of a normal standard.)

Solution: By symmetry we just need to look at even n, that is $n = 2k$.

First method: integrating by part we get

$$\begin{aligned} m_{2k} =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2k} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2k-1}\, z e^{-\frac{z^2}{2}} dz = \\ =& (2k-1) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2k-2} e^{-\frac{z^2}{2}} dz = (2k-1) m_{2k-2} \end{aligned}$$

we know that $m_0 = 1$ so that

$$m_{2k} = (2k-1)(2k-3)(2k-5)\cdots 5 \cdot 3 \cdot 1$$

Second method: We know that the moment generating function of Z is

$$M_Z(t) = e^{\frac{t^2}{2}} = \sum_{k=0}^{\infty} \frac{t^{2k}}{2^k k!}$$

so that

$$\frac{d^k}{dt^k} M_Z(t)\Big|_{t=0} = \frac{(2k)!}{2^k k!} = (2k-1)(2k-3)(2k-5)\cdots 5 \cdot 3 \cdot 1$$

5. (20 points (bonus)) Let N_1, N_2 and N_3 be discrete random variables with joint probability mass function

$$p(n_1, n_2, n_3) = \mathbb{P}(N_1 = n_1 \,\&\, N_2 = n_2 \,\&\, N_3 = n_3) = \frac{3^{-N} N!}{n_1! n_2! n_3!}$$

if $n_1 + n_2 + n_3 = N$ and 0 otherwise.

Compute the marginal mass function p_{N_1} of N_1, that is

$$p_{N_1}(n_1) = \mathbb{P}(N_1 = n_1)$$

and the conditional mass function $p_{N_2,N_3|N_1}$ of N_2 and N_3 given N_1, that is

$$p_{N_2,N_3|N_1}(n_2, n_3|n_1) = \mathbb{P}(N_2 = n_2 \,\&\, N_3 = n_3 \mid N_1 = n_1)\,.$$

(**Hint**: you can answer the question without doing any computation. Think what situation is described by N_1, N_2 and N_3.)

Solution: Observe that N_1, N_2 and N_3 are the result of repeating an experiment with 3 possible equiprobable outcomes (say 1,2,3) N times. $\mathbb{P}(N_1 = n_1)$ represents the probability of obtaining n_1 1s when the probability of a 1 in 1/3. Thus $\mathbb{P}(N_1 = n_1)$ is a binomial with $p = 1/3$ that is

$$\mathbb{P}(N_1 = n_1) = \frac{N!}{(N - n_1)! n_1!} \left(\frac{1}{3}\right)^{n_1} \left(\frac{2}{3}\right)^{N-n_1}.$$

On the other hand if you know you had exactly n_1 1's, the remaining outcomes are 2 or 3, with equal probability. Thus

$$p_{N_2,N_3|N_1}(n_2, n_3|n_1) = \frac{2^{-(N-n_1)}(N - n_1)!}{n_2! n_3!}$$

if $n_2 + n_3 = N - n_1$ and 0 otherwise.

In formulas we have

$$\begin{aligned} p_{N_1}(n_1) = \sum_{n_2,n_3} p(n_1, n_2, n_3) &= \sum_{n_2+n_3=N-n_1} \frac{3^{-N} N!}{n_1! n_2! n_3!} = \\ &= \frac{3^{-N} 2^{N-n_1} N!}{(N - n_1)! n_1!} \sum_{n_2+n_3=N-n_1} \frac{2^{-(N-n_1)}(N - n_1)!}{n_2! n_3!} = \\ &= \frac{N!}{(N - n_1)! n_1!} \left(\frac{1}{3}\right)^{n_1} \left(\frac{2}{3}\right)^{N-n_1} \end{aligned}$$

so that N_1 is a binomial r.v. with N trials and $p = 1/3$.

Moreover we have

$$p_{N_2,N_3|N_1}(n_2,n_3|n_1) = \frac{3^{-N}N!}{n_1!n_2!n_3!}\left(\frac{N!}{(N-n_1)!n_1!}\left(\frac{1}{3}\right)^{n_1}\left(\frac{2}{3}\right)^{N-n_1}\right)^{-1} = $$
$$= \frac{2^{-(N-n_1)}(N-n_1)!}{n_2!n_3!}$$

if $n_2 + n_3 = N - n_1$ and 0 otherwise.

Thus N_2 is a binomial r.v with $N - n_1$ trials and $p = 1/2$.

Useful Formulas

- **Normal Distribution**: if Z is a standard normal r.v. then its density function is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

while $E(Z) = 0$ and $V(Z) = 1$. The moment generating function $M_Z(t)$ is given by

$$M_Z(t) = e^{\frac{t^2}{2}} .$$

Moreover

$$\Phi(z) = \mathbb{P}(Z \le z)$$

is given in the table on next page. Finally if X is normal with $\mathbb{E}(X) = \mu$ and $V(X) = \sigma^2$ then

$$Y = \frac{X - \mu}{\sigma}$$

is normal standard.

- **Jensen's Inequality**: If X is a r.v. and g is a convex function then

$$\mathbb{E}(g(X)) \ge g(\mathbb{E}(X)) .$$

- **CLT**: if X_i is a sequence of i.i.d. random variable with expected value μ and variance σ^2 and

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$$

then S_n converges in distribution to a normal standard r.v. Z.

- **Convergence in Distribution**: we say that the sequence X_n converge in distribution to X if

$$\mathbb{P}(X_n \le x) \to_{n\to\infty} \mathbb{P}(X \le x)$$

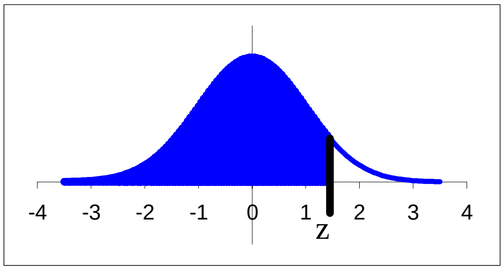
for every $x \in \mathbb{R}$.

Table 1b: Standard Normal Probabilities

The values in the table below are cumulative probabilities for the standard normal distribution *Z* (that is, the normal distribution with mean 0 and standard deviation 1). These probabilities are values of the following integral:

$$P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Geometrically, the values represent the area to the left of *z* under the density curve of the standard normal distribution:



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998