No books or notes allowed. No laptop or wireless devices allowed. Show all your work for full credit. **Write clearly and legibly**.

Name (print): ____

Question:	1	2	3	4	Total
Points:	30	40	15	15	100
Score:					

Question 1 .... *30 point*

Let X be a normal random variable with $\mu = \mathbb{E}(X) = 2$ and $\sigma^2 = \text{var}(X) = 4$.

(a) (15 points) Compute $\mathbb{P}(X < 0)$. Express the result in term of the probability integral Φ.

Solution: Standardizing we get:

$$\mathbb{P}(X < 0) = \mathbb{P}\left(\frac{X-2}{2} < -1\right) = \Phi(-1)$$

(b) (15 points) Find δ such that

$$\mathbb{P}(2 - \delta < X < 2 + \delta) = 0.95\,.$$

Express the result in term of the α critical value z_α.

Solution: In this case we obtain

$$\mathbb{P}(2 - \delta < X < 2 + \delta) = \mathbb{P}\left(-\frac{\delta}{2} < \frac{X-2}{2} < \frac{\delta}{2}\right)$$

so that we need

$$\Phi\left(-\frac{\delta}{2}\right) = 0.025$$

and

$$\delta = 2z_{0.025}.$$

Question 2 .... *40 point*

Let X and Y be two r.v. such that the marginal p.d.f. of X is

$$f_X(x) = \begin{cases} 4xe^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and the conditional p.d.f. of Y given X is

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 < y < x \\ 0 & \text{otherwise}\,. \end{cases}$$

This means that, given $X = x$, Y in uniform in $[0, x]$.

(a) (10 points) Write the joint p.d.f. $f(x, y)$ of X and Y.

Solution: Clearly we have

$$f(x, y) = \begin{cases} 4e^{-2x} & x > y > 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

(b) (15 points) Compute the marginal p.d.f. of $f_Y(y)$ of Y and the conditional p.d.f. $f_{X|Y}(x|y)$ of X given Y.

Solution:
We have

$$f_Y(y) = \int_y^\infty 4e^{-2x}dx = 2e^{-2y} \qquad y > 0$$

so that

$$f_{X|Y}(x|y) = \frac{4e^{-2x}}{2e^{-2y}} = 2e^{-2(x-y)} \qquad x > y > 0\,.$$

(c) (15 points) Compute $\mathbb{P}(Y > X/2)$.(**Hint**: consider first $\mathbb{P}(Y > X/2|X = x)$.)

Solution: Observe that

$$\mathbb{P}(Y > X/2) = \int_0^\infty \mathbb{P}(Y > X/2|X = x) f_X(x) dx$$

but

$$\mathbb{P}(Y > X/2|X = x) = \frac{1}{2}$$

since Y is uniform in $[0, x]$. Thus we have

$$\mathbb{P}(Y > X/2) = \frac{1}{2}$$

Alternatively we have

$$\mathbb{P}(Y > X/2) = \iint_{0<y<x/2} 4e^{-2x} dx\, dy = \int_0^\infty \int_0^{x/2} 4e^{-2x} dy\, dx = \int_0^\infty 2xe^{-2x} dx = \frac{1}{2}$$

Question 3 .... *15 point*

Let X and Y be two independent Normal Standard r.v., that is the joint p.d.f. of X and Y is

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} .$$

Call

$$U = X + Y$$
$$V = X - Y .$$

Compute the joint p.m.f. of U and V. Are they independent?

Solution: We first write X and Y in term of U and V has

$$X = \frac{1}{2}(U+V)$$
$$y = \frac{1}{2}(U-V) .$$

Clearly we have

$$\left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| = \frac{1}{2}$$

Since

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

we get

$$f_{X,Y}(x,y) = \frac{1}{4\pi} e^{-\frac{u^2+v^2}{4}} = \frac{1}{2\sqrt{\pi}} e^{-\frac{u^2}{4}} \quad \frac{1}{2\sqrt{\pi}} e^{-\frac{v^2}{4}}$$

and U and V are two independent Normal r.v. with expected value 0 and variance $\sqrt{2}$.

Question 4 .... *15 point*

If X is a continuous r.v., the upper quintile $q(0.8)$ of the p.d.f. of X is defined by

$$\mathbb{P}(X < q(0.8)) = 0.8.$$

A Pareto r.v. X with shape α is defined by the p.d.f.

$$f(x) = \begin{cases} \alpha x^{-(\alpha+1)} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

where $\alpha > 1$.

Compute $q(0.8)$ when X is a Pareto r.v. with shape α.

Solution: We have

$$\mathbb{P}(X \leq x) = \int_1^x \frac{\alpha}{y^{\alpha+1}} dy = x^{-\alpha} - 1$$

so that

$$q(0.8) = 0.2^{-1/\alpha}.$$

Useful Formulas

- **Exponential Distribution**: if T is an exponential r.v. with parameter λ then its density function is
$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if} \quad t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
while $E(T) = 1/\lambda$ and $F(x) = P(X \leq x) = 1 - e^{-\lambda x}$.

- **Normal distribution**: if X is a Normal random variable with $\mathbb{E}(X) = \mu$ and $\text{var}(X) = \sigma^2$ then
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$
Moreover $Z = (X - \mu)/\sigma$ is Standard Normal, that is Z is normal with $\mu = 0$ and $\sigma^2 = 1$. The c.d.f. of Z is $\Phi(x) = \mathbb{P}(Z \leq z)$ and the α-critical value z_α is defined by $\Phi(-z_\alpha) = \alpha$.

- **Uniform distribution**: If X is a Uniform r.v. in $[A, B]$ then
$$f(x) = \begin{cases} \frac{1}{B-A} & A < x < B \\ 0 & \text{otherwise}. \end{cases}$$