This is a take home midterm. You can use your notes, my online notes on canvas and the textbooks book. You are supposed to work on your own text without external help. I'll be available to answer question in person or via email. Please, write clearly and legibly and take a readable scan before uploading.

Name (print): ______________________________

Question:	1	2	3	4	5	Total
Points:	20	35	15	15	15	100
Score:						

Question 1 . . . . *20 point*

Let X be a normal random variable with $\mu = \mathbb{E}(X) = 2$ and $\sigma^2 = \mathrm{var}(X) = 4$.

(a) (10 points) Compute $\mathbb{P}(X < 0)$. Express the result in term of the cumulative distribution function Φ of a Normal Standard r.v.. Φ is usually called the *probability integral.*

Solution: Standardizing we get:

$$\mathbb{P}(X < 0) = \mathbb{P}\left(\frac{X-2}{2} < -1\right) = \Phi(-1)$$

(b) (10 points) Find δ such that

$$\mathbb{P}(2 - \delta < X < 2 + \delta) = 0.95\,.$$

Express the result in term of the α critical value z_α defined as $\Phi(-z_\alpha) = \alpha$.

Solution: In this case we obtain

$$\mathbb{P}(2 - \delta < X < 2 + \delta) = \mathbb{P}\left(-\frac{\delta}{2} < \frac{X-2}{2} < \frac{\delta}{2}\right)$$

so that we need

$$\Phi\left(-\frac{\delta}{2}\right) = 0.025$$

and

$$\delta = 2z_{0.025}.$$

Question 2 .... *35 point*

Let X and Y be two r.v. such that the marginal p.d.f. of X is

$$f_X(x) = \begin{cases} 4xe^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and the conditional p.d.f. of Y given X is

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 < y < x \\ 0 & \text{otherwise}. \end{cases}$$

This means that, given $X = x$, Y in uniform in $[0, x]$.

(a) (10 points) Write the joint p.d.f. $f(x, y)$ of X and Y.

Solution: Clearly we have

$$f(x, y) = \begin{cases} 4e^{-2x} & x > y > 0 \\ 0 & \text{otherwise}. \end{cases}$$

(b) (10 points) Compute the marginal p.d.f. of $f_Y(y)$ of Y and the conditional p.d.f. $f_{X|Y}(x|y)$ of X given Y.

Solution:

We have

$$f_Y(y) = \int_y^\infty 4e^{-2x}dx = 2e^{-2y} \qquad y > 0$$

so that

$$f_{X|Y}(x|y) = \frac{4e^{-2x}}{2e^{-2y}} = 2e^{-2(x-y)} \qquad x > y > 0.$$

(c) (15 points) Compute $\mathbb{P}(Y > X/2)$.(**Hint**: consider first $\mathbb{P}(Y > X/2|X = x)$.)

Solution: Observe that

$$\mathbb{P}(Y > X/2) = \int_0^\infty \mathbb{P}(Y > X/2|X = x) f_X(x) dx$$

but

$$\mathbb{P}(Y > X/2|X = x) = \frac{1}{2}$$

since Y is uniform in $[0, x]$. Thus we have

$$\mathbb{P}(Y > X/2) = \frac{1}{2}$$

Alternatively we have

$$\mathbb{P}(Y > X/2) = \iint_{0<y<x/2} 4e^{-2x} dx\, dy = \int_0^\infty \int_0^{x/2} 4e^{-2x} dy\, dx =$$
$$\int_0^\infty 2xe^{-2x} dx = \frac{1}{2}$$

Question 3 . . . . *15 point*

Let X and Y be two independent Normal Standard r.v., that is the joint p.d.f. of X and Y is

$$f_{X,Y}(x,y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}\,.$$

Call

$$U = X + Y$$
$$V = X - Y\,.$$

Compute the joint p.m.f. of U and V. Are they independent?

Solution: We first write X and Y in term of U and V has

$$X = \frac{1}{2}(U+V)$$
$$Y = \frac{1}{2}(U-V)\,.$$

Clearly we have

$$\left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| = \frac{1}{2}$$

Since

$$f_{X,Y}(x,y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$$

we get

$$f_{X,Y}(x,y) = \frac{1}{4\pi}e^{-\frac{u^2+v^2}{4}} = \frac{1}{2\sqrt{\pi}}e^{-\frac{u^2}{4}} \quad \frac{1}{2\sqrt{\pi}}e^{-\frac{v^2}{4}}$$

and U and V are two independent Normal r.v. with expected value 0 and variance $\sqrt{2}$.

Question 4 .... *15 point*

If X is a continuous r.v., the upper quintile $q(0.8)$ of the p.d.f. of X is defined by

$$\mathbb{P}(X \leq q(0.8)) = 0.8.$$

A Pareto r.v. X with shape α is defined by the p.d.f.

$$f(x) = \begin{cases} \alpha x^{-(\alpha+1)} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

where $\alpha > 1$.

Compute $q(0.8)$ when X is a Pareto r.v. with shape α.

Solution: We have

$$\mathbb{P}(X \leq x) = \int_1^x \frac{\alpha}{y^{\alpha+1}} dy = 1 - x^{-\alpha}$$

so that

$$q(0.8) = 0.2^{-1/\alpha}.$$

Question 5 .... *15 point*

Let X_1 and X_2 be two independent r.v. uniformly distributed in $[0,1]$. Find the p.d.f. of $Y = X_1 + X_2$.

Solution: Clearly

$$\mathbb{P}(Y < 0) = \mathbb{P}(Y > 2) = 0\,.$$

Observe that if X is uniform in $[0,1]$ so is $1 - X$. Thus

$$\mathbb{P}(X_1 + X_2 \le y) = \mathbb{P}(1 - X_1 + 1 - X_2 \le y) = \mathbb{P}(X_1 + X_2 \ge 2 - y)$$

so that it is enough to compute $F_Y(y)$ for $0 \le y \le 1$. We have

$$F_Y(y) = \int_0^y dx_1 \int_0^{y-x_1} dx_2 = \int_0^y (y - x_1)dx_1 = \frac{y^2}{2}$$

for $0 \le y \le 1$ and

$$F_Y(y) = 1 - \frac{(2-y)^2}{2}$$

for $1 \le y \le 2$. Finally the p.d.f. is

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ y & 0 \le y \le 1 \\ 2 - y & 1 \le y \le 2 \\ 0 & y > 2 \end{cases}$$