This is a take home midterm. You can use your notes, my online notes on canvas and the textbooks book. You are supposed to work on your own text without external help. I'll be available to answer question in person or via email. Please, write clearly and legibly and take a readable scan before uploading.

To solve the Exam problems, I have not collaborated with anyone nor sought external help and the material presented is the result of my own work.

Name (print): ______________________________

Question:	1	2	3	4	5	Total
Points:	10	20	30	20	20	100
Score:						

Question 1 . . . . *10 point*

Let X be a continuous r.v. such that

$$\mathbb{P}(X > 0 \,\&\, X < M) = 1$$

for some $M > 0$. Show that

$$\mathbb{E}(X) = \int_0^M (1 - F_X(x))dx$$

where F_X is the c.d.f. of X.

Solution:

Since $(1 - F_X(x))' = -f_X(x)$ we have

$$\mathbb{E}(X) = \int_0^M x f_X(x)dx = x(1 - F_X(x))\Big|_{x=0}^M + \int_0^M (1 - F_X(x))dx\,.$$

Since $F(M) = \mathbb{P}(X < M) = 1$ we get the thesis.

Alternatively observe that

$$1 - F_X(x)) = \int_x^\infty f_X(y)dy = \int_x^M f_X(y)dydx$$

so that

$$\int_0^M (1 - F_X(x))dx = \int_0^M \int_x^M f_X(y)dydx = \int_0^M \int_0^y f_X(y)dxdy =$$
$$= \int_0^M y f_X(y)dy = \mathbb{E}(X)$$

Question 2 .... *20 point*

Let X_t describe the price of a stock at time t, where time is measured in days and $t = 0, 1, 2, \ldots$. Assume that X_t satisfies:

$$X_{t+1} = \Delta_t X_t$$

where Δ_t, $t = 0, 1, 2, \ldots$, form a family of i.i.d. random variables with p.d.f.:

$$f_\Delta(\delta) = \frac{1}{\delta\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log\delta - \mu)^2}{2\sigma^2}\right) \qquad \delta \geq 0\,.$$

(a) (10 points) Assume that $X_0 = 1$ with probability 1. Find the p.d.f. of X_t for $t > 0$. (**Hint:** you can express Δ is term of a normal r.v..)

Solution: Call $Z_t = \log\Delta_t$. From the formula of change of variable we get that the p.d.f. of Z is

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z - \mu)^2}{2\sigma^2}\right)$$

that is Z is normal with $E(Z) = \mu$ and $var(Z) = \sigma^2$. Thus we can write

$$X_{t+1} = e^{Z_t} X_t$$

from which we get $X_t = e^{T_t}$ where $T_t = \sum_{s=0} t - 1 Z_s$ is a normal r.v. with $\mathbb{E}(T_t) = t\mu$ and $var(T_t) = t\sigma^2$. Changing variables back we get:

$$f_{X_t}(x) = \frac{1}{x\sqrt{2\pi t\sigma^2}} \exp\left(-\frac{(\log x - t\mu)^2}{2t\sigma^2}\right) \qquad x \geq 0\,.$$

(b) (10 points) Assume now that $\mu = 0.1$ and $\sigma^2 = 0.2$. Find $\bar{x}$ such that

$$\mathbb{P}(X_{10} > \bar{x}) = 0.75.$$

You can use a calculator like the one here.

Solution:

We have

$$\mathbb{P}(X_{10} > \bar{x}) = \mathbb{P}(T_{10} > \log(\bar{x})) = \mathbb{P}\left(\frac{T_{10} - 10\mu}{\sqrt{10}\sigma} > \frac{\log(\bar{x}) - 10\mu}{\sqrt{10}\sigma}\right) = 1 - \Phi\left(\frac{\log(\bar{x}) - 10\mu}{\sqrt{10}\sigma}\right) = 0.75\,.$$

Since $\Phi^{-1}(0.25) = -0.674$ we get

$$\bar{x} = \exp(-1.414 \cdot 0.674 + 1) = 1.048\,.$$

Question 3 .... *30 point*

Let X, Y have a joint distribution that is uniform on the unit circle in $\mathbb{R}^2$, that is

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } \; x^2 + y^2 \le 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

(a) (10 points) Let $R = X^2 + Y^2$ and Θ be the r.v. defined by

$$\cos(\Theta) = \frac{X}{\sqrt{X^2 + Y^2}} \qquad \sin(\Theta) = \frac{Y}{\sqrt{X^2 + Y^2}}$$

with $0 \le \Theta < 2\pi$. Show that R and Θ are in dependent and find the p.d.f. f_R of R and f_Θ of Θ.

Solution: By construction we have

$$X = \sqrt{R}\cos(\Theta) \qquad Y = \sqrt{R}\sin(\Theta)$$

so that the Jacobian is

$$J = \begin{pmatrix} -\frac{\cos(\theta)}{2\sqrt{r}} & -\sqrt{r}\sin(\theta) \\ -\frac{\sin(\theta)}{2\sqrt{r}} & \sqrt{r}\cos(\theta) \end{pmatrix}$$

and $|J| = 1/2$. Thus we get $f_{R,\Theta} = 1/2\pi$ for $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Thus R is uniform in $[0,1]$ while Θ is uniform in $[0, 2\pi]$ and they are independent.

(b) (10 points) Let $T = \sqrt{-2\log(X^2 + Y^2)}$. Fint the p.d.f. f_T of T.

Solution: We have

$$\mathbb{P}(T \ge t) = \mathbb{P}(X^2 + Y^2 \le e^{-t^2/2}) = e^{-t^2/2}$$

so that

$$f_T(t) = te^{-t^2/2}\,.$$

(c) (10 points) Consider the r.v.

$$U = \frac{X}{\sqrt{X^2+Y^2}}\sqrt{-2\log(X^2+Y^2)}$$
$$V = \frac{Y}{\sqrt{X^2+Y^2}}\sqrt{-2\log(X^2+Y^2)}\,.$$

Show that U, V are i.i.d normal standard r.v.

Solution: We have

$$U = \cos(\Theta)T$$
$$V = \sin(\Theta)T\,.$$

or

$$T = \sqrt{U^2+V^2}$$
$$\Theta = \arctan\left(\frac{U}{V}\right)$$

and we get $|J| = 1/\sqrt{U^2+V^2}$. Using the change of variable formula we get

$$f_{U,V}(u,v) = \frac{1}{2\pi}e^{-(u^2+v^2)/2} = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}\frac{1}{\sqrt{2\pi}}e^{-v^2/2}\,.$$

Question 4 . . . . *20 point*

Let X_i, $i = 1, 2, 3, \ldots$, be a family of i.i.d random variables with exponential distribution with parameter 1, that is

$$f_{X_i}(x) = e^{-x}$$

For $n \geq 1$ defines $T_n = \sum_{i=1}^n X_i$.

You can think that X_i is the time between the i-th and the $i+1$-th arrival of a request for service at a certain computer server. In this case, T_n is the time of arrival of the n-th request for service.

(a) (10 points) Show that the p.d.f. of T_n is

$$f_{T_n}(t) = \frac{t^{n-1}}{(n-1)!}e^{-t}$$

(**Hint**: Use change of variables to find the joint p.d.f. of the T_i, $i = 1, 2, \ldots, n$, and then compute the marginal on T_n. Alternatively you can use induction.)

Solution: The joint p.d.f of $X_1, X_2, \ldots X_n$ is $f_{X,n}(x_1, \ldots, x_n) = \exp(-\sum_{i=1}^n x_i)$ for $x_i > 0$ and 0 otherwise. Using the formula of change of variables we get that the joint p.d.f. of $T_1, T_2, \ldots, T_n$ is

$$f_{T_n}(t_1, \ldots, t_n) = \exp(-t_n)$$

if $0 < t_1 < t_2 < \cdots < t_n$ and 0 otherwise. Thus we get

$$f_{T_n}(t_n) = \int_{0<t_1<\cdots<t_n} dt_1 \cdots dt_{n-1} e^{-t_n} = t_n^{n-1} e^{-t_n} \int_{0<s_1<\cdots<s_{n-1}<1} ds_1 \cdots ds_{n-1}$$

where we called $s_i = t_i/t_n$. Since

$$\int_{0<s_1<\cdots<s_{n-1}<1} ds_1 \cdots ds_{n-1} = \frac{1}{(n-1)!}$$

we get

$$f_{T_n}(t_n) = \frac{t_n^{n-1}}{(n-1)!}e^{-t_n}\,.$$

Observe that $T_{n+1} = T_n + X_{n+1}$ and T_n is independent from X_{n+1} so that

$$f_{T_{n+1}}(t) = \int_0^\infty f_X(t-s) f_{T_n}(s) ds$$

thus assuming that f_{T_n} satisfy the thesis we get

$$f_{T_{n+1}}(t) = \int_0^t e^{-(t-s)} \frac{s^{n-1}}{(n-1)!} e^{-s} ds = \frac{e^{-t}}{(n-1)!} \int_0^t s^{n-1} ds = \frac{t^n}{n!} e^{-t}\,.$$

(b) (10 points) Let N_t be the number of arrivals before time t, that is

$$N_t = \max\{n \,|\, T_n < t\}\,.$$

Show that N_t is a Poisson r.v. with expected value t. (**Hint:** express $\mathbb{P}(N_t < n)$ in term of T_n.)

Solution: Observe that $\mathbb{P}(N_t < n) = \mathbb{P}(T_n \geq t)$ so that

$$\mathbb{P}(N_t \leq n) = \int_t^\infty \frac{s^n}{n!}e^{-s}ds = \int_0^\infty \frac{(t+s)^n}{n!}e^{-(t+s)}ds = \sum_{k=0}^{n}\binom{n}{k}\frac{t^k e^{-t}}{n!}\int_0^\infty s^{n-k}e^{-s}ds = \sum_{k=0}^{n}\frac{t^k e^{-t}}{k!}$$

so that

$$\mathbb{P}(N_t = n) = \mathbb{P}(N_t \leq n) - \mathbb{P}(N_t \leq n-1) = \frac{t^n e^{-t}}{n!}$$

Question 5 . . . . *20 point*

Two question on continuous/discrete r.v.

(a) (10 points) Let X be continuous r.v. uniformly distributed in $[0, 1]$. Consider the r.v.

$$Z = \max\{X, 0.5\}\,.$$

Find the c.d.f. of Z. Is Z a continuous r.v.? Is it discrete?

Solution: Clearly we have $\mathbb{P}(Z < 0.5) = 0$ while $\mathbb{P}(Z \leq z) = \mathbb{P}(X \leq z)$ if $z \geq 0.5$ so that we have

$$F_Z(z) = \begin{cases} 0 & z < 0.5 \\ z & 0.5 \leq z < 1 \\ 1 & z \geq 1 \end{cases}$$

Z is not a continuous r.v. since F_Z is not continuous and it is not discrete since F_Z is not piecewise constant.

(b) (10 points) Let X be continuous r.v. with p.d.f. f_X and Y be a discrete r.v. with p.m.f. p_Y. Moreover X and Y are independent. Find the p.d.f. of $Z = X + Y$. Is Z a continuous r.v.?

Solution: We have

$$\mathbb{P}(Z \leq z) = \sum_y \mathbb{P}(X \leq z - y \,\&\, Y = y) = \sum_y \mathbb{P}(X \leq z - y)\mathbb{P}(Y = y)$$

so that

$$f_Z(z) = \sum_y f_X(z - y)p_Y(y)$$

In general we cannot say whether it is continuous but if Y takes only finitely many values than Z is continuous.