Exercises

2.10

$I_E(\omega)$ takes only two values, 0 and 1. By definition

$$\{\omega \mid 1_E(\omega) = 1\} = E \in \mathcal{F}$$

and

$$\{\omega \mid 1_E(\omega) = 0\} = \Omega \setminus E \in \mathcal{F}$$

2.11

We have

$$\{\omega \mid U(\omega) = 1\} = \{1\} \notin \mathcal{F}$$

Thus U is not a discrete random variable
The same argument shows that W is not a discrete random variable.

Finally $V = 1_E$ where

$$E = \{2, 4, 6\} \in \mathcal{F}$$

so that V is a discrete random variable.

2.24

$$\mathbb{P}(X>k) = \sum_{i=k+1}^{\infty} pq^{i-1} = pq^{k}\sum_{i=0}^{\infty} q^{i} =$$

$$= \frac{p}{1-q}\, q^{k} = (1-p)^{k}$$

2.38

$$\mathrm{var}(aX+b) = \mathbb{E}((aX+b)^2) - \mathbb{E}(aX+b)^2$$

we know that

$$\mathbb{E}(aX+b) = a\mathbb{E}(X) + b$$

on the other hand we have

$$\mathbb{E}((aX+b)^2) = \mathbb{E}(a^2X^2 + 2abX + b^2) =$$

$$= a^2\mathbb{E}(X^2) + 2ab\mathbb{E}(X) + b^2$$

We thus get

$$\mathrm{var}(aX+b) = a^2\mathbb{E}(X^2) - a^2\mathbb{E}(X)^2 =$$

$$= a^2 \cancel{\mathbb{E}}\, \mathrm{var}(X)$$

Problems

4. We need that

$$\sum_{k=1}^{\infty} c k^{\alpha} = 1$$

Thus it must be

$$c = \frac{1}{\sum_{k=1}^{\infty} k^{\alpha}}$$

For this to make sense we need

$$\sum_{k=1}^{\infty} k^{\alpha} < +\infty \Rightarrow \qquad \alpha < -1$$

So that $p(k)$ is a mass function if

$$\alpha < -1 \quad \text{and} \quad c = \left(\sum_{k=1}^{\infty} k^{\alpha}\right)^{-1}$$

5. We saw that

$$\mathbb{P}(X > k) = (1-p)^k$$

so that

$$\mathbb{P}(X > m+n \mid X > m) =$$

$$\frac{\mathbb{P}(X > m+n \text{ and } X > m)}{\mathbb{P}(X > m)} = (*)$$

$$= \frac{\mathbb{P}(X > m+n)}{\mathbb{P}(X > m)} = \frac{(1-p)^{m+n}}{(1-p)^{m}} = (1-p)^{n} =$$

$$= \mathbb{P}(X > n)$$

where in $(*)$ we have used that

$$\{X > m+n\} \subset \{X > m\}.$$

Assume now that X satisfies

$$\mathbb{P}(X > m+n \mid X > m) = \mathbb{P}(X > m)$$

It follows that

$$\mathbb{P}(X > m+1 \mid X > m) = \mathbb{P}(X > 1)$$

Call $\mathbb{P}(X > 1) = q$ has before we get

$$\mathbb{P}(X > m+1) = q\,\mathbb{P}(X > m)$$

Since $\mathbb{P}(X > 0) = 1$ we get, by recursion

$$\mathbb{P}(X > m) = q^{m}$$

or

$$\mathbb{P}(X = m) = \mathbb{P}(X > m-1) - \mathbb{P}(X > m) = q^{m-1} - q^{m} =$$

$$= p\,q^{m-1}.$$

7) We can use inclusion/exclusion. Let

A_i The event That The first n coupons collected do not contain coupon i.
We have

$$\mathbb{P}(A_i) = \frac{(c-1)^n}{c^n}$$

Clearly $A_i \cap A_j$ is The event That the first n coupons do not contain i and j.
Thus

$$\mathbb{P}(A_i \cap A_j) = \frac{(c-2)^n}{c^n}$$

Similarly

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \dots A_{i_k}) = \frac{(c-k)^n}{c^n}$$

Thus the probability we look for is

$$\mathbb{P}(\bigcup_i A_i) = \binom{c}{1}\frac{(c-1)^n}{c^n} - \binom{c}{2}\frac{(c-2)^n}{c^n} + \dots$$

$$(-1)^c \binom{c}{c-1} \frac{1}{c^n}$$

where The binomial factors $\binom{c}{i}$ count The number of subset of size i in The group of c coupons. ~~Thus~~

~~8)~~

Let N be the random variable That is n if you have a complete set when you get the n-th coupon but you did not have it before. Than

$$\mathbb{P}(N > k) = \sum_{i=1}^{c} (-1)^{i+1} \binom{c}{i} \frac{(c-i)^k}{c^k}$$

From 2.6.6 we get that

$$\mathbb{E}(N) = \sum_{n=0}^{\infty} \sum_{i=1}^{c} (-1)^{i+1} \binom{c}{i} \left(\frac{c-i}{c}\right)^k =$$

$$= \sum_{i=1}^{c} \binom{c}{i} (-1)^{i+1} \frac{c}{i}$$

Alternatively you can say that with the first selection will give you a coupon of type c_1. The arrival of the next coupon different from c_1 is a geometric r.v. with $p = \frac{c-1}{c}$ because there are $c-1$ coupon different from c_1. Thus it takes in average $1 + \frac{c}{c-1}$ coupons to get 2 different ones, say c_1 & c_2. Similarly the arrival of the next coupon different from c_1 & c_2 will in average take $\frac{c}{c-2}$ selection so that we have

$$\mathbb{E}(N) = \sum_{i=0}^{c-1} \frac{c}{c-i}$$

Thus we have

$$\sum_{i=0} \binom{c}{i} (-1)^{i+1} \frac{c}{i} = \sum_{i=0}^{c-1} \frac{c}{c-i}.$$

This can be shown directly.