Chapter 8.1 n 2 :

$Y = \overline{X}_n - \theta$ is a Normal r.v. with $\mathbb{E}(Y) = 0$ and $Var(Y) = 4/n$. Thus we have that

$$Z = \frac{\sqrt{n}}{2}\left(\overline{X}_n - \theta\right)$$

is Standard Normal. Thus

$$\mathbb{E}(|Y|^2) < 0.1$$

$$\Downarrow$$

$$\frac{2}{\sqrt{n}} \mathbb{E}(|Z|^2) < 0.1$$

$$\Downarrow$$

$$n \geq 400$$

n 9

The MLE of θ is

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} X_i}$$

We know that $\sum_{i=1}^{n} X_i$ has a Gamma distribution with par. n and θ. Calling $\Gamma(\cdot)$ the c.d.f of a Gamma r.v. with par n and θ we get

$$\mathbb{P}(\hat{\theta} \leq t) = \mathbb{P}\left(\sum_{i=1}^{n} X_i \geq t/n\right) =$$

$$= 1 - \Gamma(t/n)$$

Chapter 8.2 n 4

We want to find r such that

$$\mathbb{P}(X^2 + Y^2 \le r) = 0.99$$

Since $X^2 + Y^2$ has a χ^2_2 distribution

we get

$$r = \chi_2^{-1}(0.99) = 9.21$$

n. 7

Observe that

$$T_i = F_i(Y_i)$$

is a uniform r.v. in $[0,1]$.

Indeed

$$\mathbb{P}(T_i \le z) = \mathbb{P}(F_i(Y_i) \le z) =$$

$$\mathbb{P}(Y_i \le F_i^{-1}(z)) = F_i(F_i^{-1}(z)) = z.$$

With a similar computation

it follows that

$$Z_i = -2 \log F_i(X_i)$$

is a χ^2_2 r.v.

Finally This implies That

$$Y = \sum_i Z_i$$

is a χ^2_{2n} r.v.

n 10

Observe That

$$X_1 + X_2 + X_3$$

is normal with $Var(Z_1) = 3$.

It follows That

$$Z_1 = \frac{1}{\sqrt{3}} (X_1 + X_2 + X_3)$$

is standard normal.

Similarly

$$Z_2 = \frac{1}{\sqrt{3}} \left(X_4 + X_5 + X_6 \right)$$

is standard normal.

Thus

$$\frac{1}{3} Y = Z_1^2 + Z_2^2$$

is χ^2_2 so that $c = \frac{1}{3}$

Chapter 8.3 n 5

Since X_1 and X_2 form a bivariate normal r.v., also $Z_1 = X_1 + X_2$ and $Z_2 = X_1 - X_2$ form a bivariate normal r.v.

Moreover

$$Cov(Z_1, Z_2) = Var(X_1) - V(X_2) = 0$$

so that Z_1 and Z_2 are independent

n 7

Since

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2$$

we have that

$$n \frac{\hat{\sigma}^2}{\sigma^2} \quad \text{is} \quad \chi^2_{n-1}$$

and

a)
$$\mathbb{P}\left(\frac{\widehat{\sigma}^2}{\sigma^2} \le 1.5\right) = \mathbb{P}\left(\chi^2_{n-1} \le 1.5\, n\right)$$

From The Table we get

$$\mathbb{P}\left(\chi^2_{19} \le 30\right) < 0.95$$

$$\mathbb{P}\left(\chi^2_{20} \le 31.5\right) > 0.95$$

We have $n = 21$.

b)
$$\mathbb{P}\left(|\widehat{\sigma}^2 - \sigma^2| \le \frac{1}{2}\sigma^2\right) =$$

$$\mathbb{P}\left(|\chi^2_{n-1} - n| \le \frac{n}{2}\right) =$$

$$\mathbb{P}\left(\chi^2_{n-1} \le \frac{3}{2} n\right) - \mathbb{P}\left(\chi^2_{n-1} \le \frac{1}{2} n\right)$$

again we get

$n = 13$