MATH 3670 First Midterm February 17, 2016

No books or notes. No cellphone or wireless devices. Write clearly and show your work for every answer.

Name: ______________________________

Question:	1	2	3	4	Total
Points:	30	20	20	40	110
Score:					

1. The following numbers x_i, $i = 1, \ldots, 17$, represent a sample of size $n = 17$ from a given population.

6.46	6.33	6.93	7.12	7.26
11.12	7.03	6.96	5.48	5.35
6.49	7.87	6.02	6.75	5.67
4.01	5.82			

(a) (10 points) Compute the sample median and fourth spread.

Solution:
After ordering the data you obtain

4.01	5.35	5.48	5.67	5.82
6.02	6.33	6.46	6.49	6.75
6.93	6.96	7.03	7.12	7.26
7.87	11.12			

so that:

$$\tilde{x} = 6.49$$
$$lf = 5.82 \qquad uf = 7.03$$
$$fs = 7.03 - 5.82 = 1.21$$

(b) (10 points) Knowing that $\sum_{i=1}^{17} x_i = 112.67$ and $\sum_{i=1}^{17} x_i^2 = 781.35$ compute the sample mean and variance.

Solution:

$$\bar{x} = \frac{112.67}{17} = 6.63$$
$$s^2 = \frac{1}{16}\left(781.35 - \frac{112.67^2}{17}\right) = 2.16$$

(c) (10 points) After checking for outlier, draw a box plot of the data.

Solution: Observe that $4.01 > 5.82 - 1.5 \cdot 1.21 = 4.005$ so that there are no outlier on the lower part of the sample. On the upper part we have $11.12 > 7.03 + 3 \cdot 1.21 = 10.66$ while $7.87 < 7.03 + 1.5 \cdot 1.21 = 8.845$ so that 11.12 is the only outlier and it is an extreme outlier.

2. Let A, B and C be three events.

   (a) (10 points) Assume that A, B and C are mutually independent. Show that A and $B \cap C$ are independent.

   Solution: We need to show that

   $$P(A \cap (B \cap C)) = P(A)P(B \cap C)$$

   Observe that by hypothesis

   $$P(A \cap (B \cap C)) = P(A \cap B \cap C) = P(A)P(B)P(C) \qquad P(B \cap C) = P(B)P(C)$$

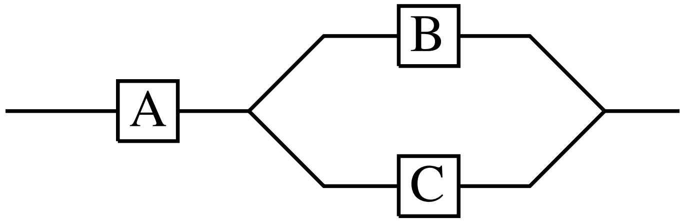
   from which the thesis follows immediately.

   (b) (10 points) Flip a fair coin twice and consider the three events A={*the first flip is Head*}, B={*the second flip is Head*}, and C={*exacly one flip is Head*}. Show that A and B are independent, A and C are independent and that B and C are independent. Are A, B and C independent?

   Solution: Observe that $P(A) = P(B) = P(C) = 0.5$. Moreover $A \cap B$={*both flip are Head*} while $A \cap C$={*the first flip is Head and the second is Tail*} and $A \cap C$={*the first is Tail and the second flip is Head*}. Thus $P(A \cap B) = P(B \cap C) = P(A \cap C) = 0.25$. Thus A and B are independent, A and C are independent and B and C are independent.

   Finally $A \cap B \cap C = \emptyset$ so that $P(A \cap B \cap C) = 0$. Thus A, B and C are not independent.

3. Consider the circuit depicted in the figure.



You know that the entire circuit work if and only if element A and at least one between elements B and C work. Assume that the probability that A works is 0.9 while the probability that B works is 0.7 and the probability that C works is 0.7. Each element works or does not work independently from the others.

(a) (10 points) Compute the probability that the entire circuit works.

Solution: Call A, B and C the events that element A, B or C (respectively) works. Call I the event the the entire circuit works. We have

$$P(I) = P(A)P(B \cup C) = 0.9P(B \cup C)$$

Observe that

$$\begin{aligned} P(B \cup C) &= 1 - P(B' \cap C') = 1 - P(B')P(C') = \\ &= 1 - (1 - P(B))(1 - P(C)) = 1 - 0.3 \cdot 0.3 \end{aligned}$$

Thus we have

$$P(I) = 0.9(1 - 0.3^2) = 0.819$$

(b) (10 points) What is the probability that element A does not work given that the circuit does not work?

Solution: We have

$$P(A'|I') = \frac{P(I' \cap A')}{P(I')} = \frac{1 - P(A)}{1 - P(I)} = \frac{0.1}{1 - 0.9(1 - 0.3^2)} = 0.552$$

4. In a bucket there are 1000 red balls and 2000 blue balls. You extract 1 ball and then flip a fair coin. If the result is Head you reinsert the ball in the bucket. If the result is Tail you keep it. You repeat the above procedure 20 times. Call N the number of balls you have kept at the end of the 20 extractions and R the number of red balls among them.

   (a) (10 points) Recognize what kind of r.v. is N and write its p.m.f.. (**Hint**: relate N with the number of Tail you got.)

   Solution:
   N is a Binomial r.v. with parameter 20 and 0.5. Thus

   $$P(N = n) = \binom{20}{n} 0.5^n 0.5^{20-n} = \binom{20}{n} 0.5^{20}$$

   (b) (10 points) What is the probability that $R = 3$ given that $N = 10$, that is find $P(R = 3 \mid N = 10)$? (**Hint:** once you know the value of N what kind of r.v. is R?)

   Solution: If $N = 10$ then R is an hypergeometric r.v with parameter 10, 1000, 2000 so that

   $$P(R = 3 \mid N = 10) = \frac{\binom{1000}{3}\binom{2000}{7}}{\binom{3000}{10}}$$

(c) (10 points) Find an expression for the probability that $R = 3$.(**Hint:** use the above results and the law of total probability.)

Solution: We have

$$P(R=3) = \sum_{n=3}^{20} P(R=3 \,|\, N=n) P(N=n) = \sum_{n=3}^{20} 0.5^{20} \frac{\binom{20}{n}\binom{1000}{3}\binom{2000}{n-3}}{\binom{3000}{n}}$$

(d) (10 points) (**Bonus**) Observing that 20 << 1000 and 20 << 2000, find an approximated p.m.f for R. (**Hint**: what is the probability that you extract a red ball and you keep it?)

Solution: We can assume that each ball, upon extraction, is red with probability $\frac{1000}{3000} = \frac{1}{3}$ and blue with probability $\frac{2000}{3000} = \frac{2}{3}$. Moreover we can assume that different extractions are independent.

After flipping the coin we have 3 possible outcomes: the ball is reinserted (with probability $\frac{1}{2}$), the ball is kept and is blue (with probability $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$) or the ball is kept and is red (with probability $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$).

Thus we can say that R is approximately a binomial r.v. with parameters 20 and $\frac{1}{6}$, that is

$$P(R=r) \simeq \binom{20}{r} \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{20-r}$$

Useful Formulas

- If A and B are events then the probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

- If X is a binomial r.v. with parameters N and p then

$$bin(x; N, p) = P(X = x) = \binom{N}{x} p^x (1-p)^{n-x}.$$

- If X is an hypergeomethic r.v. with parameters N,M and n then

$$h(x; n, N, M) = \frac{\binom{M}{x}\binom{M}{n-x}}{\binom{N+M}{n}}.$$

- If X is a Poisson r.v. with parameter λ then

$$p(x; \lambda) = P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}.$$