1) The temperature variation from one day to another is found to be distributed normally with expected value 3F and variance 3F when temperature are measured in Fahrenheit. Recalling that x Fahrenheit are equivalent to $y = 5(x-32)/9$ Celsius what is the probability distribution of the temperature variation when measured in Celsius.

Call ΔX tha variation of temperature in Fahrenheit and ΔY the variation in Celsius. Given the transormation formula between Fahrenheit and Celsius we know that a variation of Δx Fahrenheit is equivalent to a variation of $\Delta y = 5\Delta x/9$ Celsius, i.e. $\Delta Y = 5\Delta X/9$. This means that:

$$E(\Delta Y) = \frac{5}{9}E(\Delta X) = \frac{5}{3} \qquad V(\Delta Y) = \left(\frac{5}{9}\right)^2 V(\Delta X) = \frac{25}{27}$$

so that

$$\Delta Y \simeq N\left(\frac{5}{3}, \frac{25}{27}\right).$$

2) Two machines produces produce the same pill for a pharmaceutical company. Call X_1 the amount of active substance in the pills produced by the first machine and X_2 the amount of active substance in the pills produced by the second machine. After a quality review it is found that X_1 and X_2 are distributed normally with $X_1 \simeq N(1.05, 0.01)$ and $X_2 \simeq N(0.98, 0.001)$. Assuming that the two machines produce the same amount of pills and that the pill are randomly mixed before being shipped out what is the probability distribution of the amount of active substance in the pills shipped

Let A_1 be the event {*the pill was produced by the first machine*} and A_2 the event {*the pill was produced by the second machine*} and call X the amount of active substance in the pills shipped. For every $a < b$ we have:

$$P(a < X < b) = P(a < X < b|A_1)P(A_1) + P(a < X < b|A_2)P(A_2).$$

But for $i = 1, 2$ we have $P(A_i) = 0.5$ and $P(a < X < b|A_i) = P(a < X_i < b)$ so that

$$P(a < X < b) = 0.5P(a < X_1 < b) + 0.5P(a < X_2 < b).$$

If $f(x)$ is the p.d.f. of X and $f_i(x)$ the p.d.f. of X_i we then have

$$\int_a^b f(x)dx = 0.5\int_a^b f_1(x)dx + 0.5\int_a^b f_2(x)dx = \int_a^b 0.5(f_1(x) + f_2(x))dx$$

so that

$$f(x) = 0.5(f_1(x) + f_2(x)).$$

3) To be acceptable the pills must have an amount of active substance between 0.99 and 1.01. Which of the two machines has an higher probability of producing an acceptable pill.

We have to compute $P(0.99 < X_1 < 1.01)$ and $P(0.99 < X_2 < 1.01)$. We can write $X_1 = 0.1Y_1 + 1.05$ with $Y \simeq N(0,1)$ so that

$$P(0.99 < X_1 < 1.01) = P(0.99 < 0.1Y_1 + 1.05 < 1.01) =$$
$$P(-0.6 < Y_1 < -0.4) = \Phi(-0.4) - \Phi(-0.6) = 0.0703$$

In the same way we can write $X_1 = \sqrt{0.001}Y_2 + 0.98$ with $Y \simeq N(0,1)$ so that

$$P(0.99 < X_2 < 1.01) = P(0.99 < \sqrt{0.001}Y_2 + 0.98 < 1.01) =$$
$$P\left(\frac{0.01}{\sqrt{0.001}} < Y_2 < \frac{0.03}{\sqrt{0.001}}\right) = \Phi\left(\frac{0.01}{\sqrt{0.001}}\right) - \Phi\left(\frac{0.03}{\sqrt{0.001}}\right) = 0.20$$

so that the second machine has an higher probability of producing an acceptable pill.

4) If after being shipped one pill is checked and found having an amount of active substance greater that 1.05 what is the probability that it was produced by the first machine.

We want to know $P(A_1|X > 1.05)$. This given by:

$$P(A_1|X > 1.05) = \frac{P(X > 1.05|A_1)P(A_1)}{P(X > 1.05)}$$

We know that $P(A) = 0.5$ and $P(X > 1.05|A_1) = P(X_1 > 1.05) = 0.5$ (by symmetry or using the tables). Moreover

$$P(X > 1.05) = P(X > 1.05|A_1)P(A_1) + P(X > 1.05|A_2)P(A_2) =$$
$$0.5P(X_1 > 1.05) + 0.5P(X_2 > 1.05)$$

so that we must compute $P(X_2 > 1.05)$. This is

$$P(X_2 > 1.05) = P(\sqrt{0.001}Y_2 + 0.98 > 1.05) = P\left(Y_2 > \frac{0.07}{\sqrt{0.001}}\right) =$$
$$1 - \Phi\left(\frac{0.07}{\sqrt{0.001}}\right) = 0.0136$$

so that

$$P(A_1|X > 1.05) = \frac{0.5 \cdot 0.5}{0.5 \cdot 0.5 + 0.5 \cdot 0.103} = 0.987$$

5) At a bus stop a new passenger arrives every minute. The arrival time for the bus is distributed exponentially with parameter $\lambda = 0.1$. The bus can carry only 10 passenger. What is the probability that there will be passenger left at the bus stop after the bus arrived.

Let X be the r.v. that gives the arrival time of the bus. It the bus arrives after 11 minutes ore there will be at least one passenger that cannot get in so that the required probability is

$P(X > 11) = 1 - F(11)$ where $F(x)$ is the c.d.f. of an exponential variable with parameter 0.1, i.e. $F(x) = 1 - e^{-0.1x}$. We get that

$$P(X > 11) = e^{-0.1 \cdot 11} = e^{-1.1}$$

6) If X is an exponential r.v. with parameter λ compute the median of X. Which is larger, the median or the average?

Let $F(x)$ be the c.d.f. of the r.v., we know that $F(x) = 1 - e^{-\lambda x}$. The median m is defined by the equation $F(m) = 0.5$ so that we have

$$1 - e^{-\lambda m} = 0.5 \rightarrow e^{-\lambda m} = 0.5 \rightarrow m = \frac{\ln(2)}{\lambda}$$

The average of X is $\bar{m} = \frac{1}{\lambda}$ so that the median is greater than the average.

7) In a room you have 10 independent bulbs each of which has a life time distributed exponentially with parameter 0.1 when the time is measured in days. What is the probability distribution of the number of working bulbs after 10 days.

Let X_i be the breaking time of the i-th bulb. After 10 days the probability that the i-th bulb is still working is $P(X_i > 10) = 1 - F(10) = e^{-1}$. All the bulbs are working or broken independently from each other so that the distribution is binomial with parameter 10 and e^{-1}, i.e. if N is the number of bulbs working we have $N \simeq B(10, e^{-1})$.

8) A person works to install the operating system on a computer and a second one configure it after installation. Calling X the time needed by the first worker we know that it is distributed exponentially with parameter 1 when the time is measured in hours. If the first worker needed x hours to complete his job we know that the second one will need a time Y that is distributed exponentially with parameter $0.5x$. Write the joint p.d.f. of X and Y. Write an expression for the expected value of the total time needed to complete the installation and configuration. Can you compute it?

We have that the p.d.f of X is $f_X(x) = e^{-x}$. Moreover we have that the conditional p.d.f. of Y given $X = x$ is $f_Y(y|x) = 0.5xe^{-0.5xy}$. If $f(x, y)$ is the joint p.d.f. of X and Y then $f_X(x)$ is the marginal of $f(x, y)$ with respect to x. This implies that

$$f(x, y) = f_X(x)f_Y(y|x) = e^{-x}0.5xe^{-0.5xy} = 0.5xe^{-(0.5y+1)x}$$

The expected value of the total time needed to complete the installation and configuration is given by

$$E(X + Y) = \int_0^\infty \int_0^\infty (x + y)0.5xe^{-(0.5y+1)x} dxdy$$

9) Four independent dices are rolled. Let X_i be the outcome of the i-th dice and let $Y = X_1 + X_2$, $Z = X_3 + X_4$. Are Y and Z independent? Why?

The variable Y depends only on X_1 and X_2 while the variable Z depends only on X_3 and X_4. But X_1,X_2, X_3 and X_4 are independent so that Y and Z are independent.