Name: ______________________________

Question:	1	2	3	4	Total
Points:	30	10	40	20	100
Score:					

Question:	1	2	3	4	Total
Bonus Points:	10	10	0	0	20
Score:					

Question 1 .... *30 point*

A matrix U is called unitary iff

$$UU^* = I$$

where U^* denotes the transpose of U and I is the identity matrix. A matrix A is called antisymmetric iff

$$A^* = -A.$$

(a) (10 points) Show that if U is unitary then

$$\det U = \pm 1.$$

(b) (10 points) Show that if A is antisymmetric than $U = \exp(A)$ is unitary.

(c) (10 points) Show that if A is antisymmetric and $P(t)$ is a solution of

$$\dot{P}(t) = AP(t)$$

with $P(0) = I$ then $P(t)$ is unitary for every t.

(d) (10 points (bonus)) Suppose now that $P(t)$ is a solution of the equation

$$\dot{P}(t) = A(t)P(t)$$

with $P(0) = I$ and $A(t)$ antisymmetric for every t. Show that $P(t)$ is unitary for every t.(**Hint**: consider $O(t) = P(t)P(t)^*$. Show that, if $P(t)$ is unitary than $\dot{O}(t) = 0$.)

Question 2 .... *10 point*

Consider the systems of equations

$$\begin{cases} \dot{x_1} &= ax_1 + x_2 + x_1(x_1^2 + x_2^2) \\ \dot{x_2} &= -x_1 + ax_2 + x_2(x_1^2 + x_2^2) \end{cases} \tag{1}$$

with $a \leq 0$, and

$$\begin{cases} \dot{x_1} &= bx_1 + x_2 \\ \dot{x_2} &= -x_1 + bx_2 \end{cases} \tag{2}$$

with $b \leq 0$.

(a) (10 points) For which values of a and b are eq.(1) and eq.(2) locally conjugated around the point $X^* = (0,0)$? (**Hint**: linearize eq.(1) and then use the Theorems at page 168 and 66 of the book.)

(b) (10 points (bonus)) Is the conjugacy global? (**Hint**: eq.(1) has a periodic orbit around 0 for $a \leq 0$.)

Question 3 . . . . *40 point*

Consider the differential equations:

$$\begin{cases} \dot{x_1} &= 1 - \frac{x_1^2}{x_1^2+x_2^2} - Bx_2 \\ \dot{x_2} &= -\frac{x_1 x_2}{x_1^2+x_2^2} + Bx_1 \end{cases} \tag{3}$$

where B is a parameter.

(a) (10 points) Calling

$$E(X) = x_1^2 + x_2^2$$

show that if $X(t) = (x_1(t), x_2(t))$ is a solution of eq.(3) than $E(X(t))$ is constant in time, *i.e.* $\dot{E}(X(t)) = 0$.

(b) (10 points) write $(x_1, x_2) = (r\cos\theta, r\sin\theta)$ and rewrite eq.(3) as a system of equations for r and θ. Observe that, from point (a) the equation for r is simply $\dot{r} = 0$.

(c) (10 points) Consider the equation for θ with B as a parameter and $r = 1$. Find the fixed points as a function of B. Draw the bifurcation diagram and describe the bifurcations you encounter as B varies.

Question 4 . . . . *20 point*

Consider the differential equation

$$\dot{X} = \begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} X$$

Write the general solution for every value of a.