Fall 05
Math 4581

Name: ______________________________

Final

Bonetto

1) The equation governing the temperature in a rod of length 1 is:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} + c(T - u(x,t)) \qquad 0 < x < 1 \\ u(0,t) = T \\ \dfrac{\partial u(1,t)}{\partial x} = \Phi \\ u(x,0) = T \end{cases}$$

a) Find the steady state for this equation and write the equation for the difference $w(x,t) = u(x,t) - v(x)$.

b) Use separation of variables to write the general solution of the equation.

c) Using that

$$\int e^{kx}\sin(\lambda x)dx = \frac{e^{kx}\left(k\sin(\lambda x) - \lambda\cos(\lambda x)\right)}{k^2+\lambda^2}$$
$$\int e^{kx}\cos(\lambda x)dx = \frac{e^{kx}\left(k\cos(\lambda x) + \lambda\sin(\lambda x)\right)}{k^2+\lambda^2}$$

find the solution of the equation with the given initial condition.

d) Assume now that the conductivity depend on x. The equation becomes:

$$\begin{cases} p(x)\dfrac{\partial u(x,t)}{\partial t} &= \dfrac{\partial^2 u(x,t)}{\partial x^2} + c(T - u(x,t)) \\ u(0,t) &= T \\ \dfrac{\partial u(1,t)}{\partial x} &= \Phi \\ u(x,0) &= T \end{cases}$$

Use separation of variable to reduce the problem to a Sturm-Liouville problem. Write the solution of the the above equation using the solution of the Sturm Louiville problem (without finding them) and write an equation for the coefficient.

2) A square membrane of side 1 made of an anisotropic material vibrates freely with the boundary held fixed. The equation is thus

$$\begin{cases} \dfrac{\partial^2 u(x,y,t)}{\partial t^2} =& \dfrac{\partial^2 u(x,y,t)}{\partial x^2} + 2.5\,\dfrac{\partial^2 u(x,y,t)}{\partial y^2} \\ u(0,y,t) =& u(1,y,t) = u(x,0,t) = u(x,1,t) = 0 \\ u(x,y,0) =& f(x,y) \\ \dfrac{\partial u(x,y,0)}{\partial t} =& g(x,y) \end{cases}$$

where $0 < x < 1$ and $0 < y < 1$.

a) Use separation of variables to write the equation as an equation on t and one on x,y (eigenvalues equation). Write the general solution of the equation on t.

b) Use separation of variables again to write the eigenvalues equation as two Sturm-Liouville problems, one for x and one for y. State clearly the relation between the solutions of these two Sturm-Liouville problems and those of the initial eigenvalues equation.

c) Find the solution of the Sturm-Liouville problems of point b).

d) Write the lowest 5 natural frequencies of the membrane. Remember that if $f(t) = \cos(\omega t)$ than the frequency of f is $\omega/2\pi$.

e) Using the results of a), b) and c) write the general solution of this equation.

3) A string of length 2 is governed by the equation

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = \dfrac{\partial^2 u(x,t)}{\partial x^2} \\ u(0,t) = u(2,t) = 0 \\ u(x,0) = f(x) \\ \dfrac{\partial u(x,0)}{\partial t} = g(x) \end{cases}$$

where

$$f(x) = \begin{cases} 2x - 2(1-a) & 1-a < x < 1 \\ -2x + 2(1+a) & 1 < x < 1+a \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = 0$$

and $a = 0.2$. Sketch the solution of the equation for $t = 0.1$, 0.4, 1 and 1.4. Write the analytic form of the solution for $t = 0.4$.

Let now the initial condition be

$$f(x) = \begin{cases} x - (1-2a) & 1-2a < x < 1-a \\ -x+1 & 1-a < x < 1 \\ x-1 & 1 < x < 1+a \\ -x+(1+2a) & 1+a < x < 1+2a \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} 1 & 1-2a < x < 1-a \\ -1 & 1-a < x < 1 \\ 1 & 1 < x < 1+a \\ -1 & 1+a < x < 1+2a \\ 0 & \text{otherwise} \end{cases}$$

where, as before, $a = 0.2$. Find and sketch the solution at $t = 0.2$. (**Hint**: relate the initial conditions to the previous ones).