Fall 05 Name: _____

Math 4581 Test 2 Bonetto

1) The equation governing the temperature $u(x,t)$ inside a rod is:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} & 0 \le x \le 1 \\ \dfrac{\partial u(0,t)}{\partial x} = r u(0,t) \\ \dfrac{\partial u(1,t)}{\partial x} = r\left(T - u(1,t)\right) \\ u(x,0) = x \end{cases}$$

a) write and solve the equation for the steady state $v(x)$.

b) write the equation for the difference $w(x,t) = u(x,t) - v(x)$.

c) use separation of variable to reduce the problem to a Sturm-Luiville problem. Find the eigenvalues and eigenfunctions. Explain why you can expand in eigenfunctions. Write the general solution for $w(x,t)$ and an expression for the coefficient in term of $w(x,0)$.

e) Give an estimate from above and below of the first eigenvalue. How long do you have to wait to be sure that $|w(x,t)| \leq 10^{-3}$. Use only the series truncated at the first term and give the estimate as a function of the first coefficient of $w(x,t)$, *i.e.* do not try to compute it.

f) **Bonus**: write the solution of the problem. Remember that

$$\int x\cos(\lambda x)dx = \frac{\cos(\lambda x)}{\lambda^2} + \frac{x\sin(\lambda x)}{\lambda}$$
$$\int x\sin(\lambda x)dx = \frac{\sin(\lambda x)}{\lambda^2} - \frac{x\cos(\lambda x)}{\lambda}$$

2) Let $f(x)$ a continuous and differentiable function defined for all x. Assume that

$$|f(x)| \leq Ce^{-\lambda|x|}$$

with C and λ positive. Finally let

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} f(x)dx. \tag{1}$$

Consider now the function

$$F(x) = \sum_{n=-\infty}^{\infty} f(x + nL)$$

with $L > 0$.

a) Show that $F(x)$ exists and it is periodic of period L.

b) Let

$$F(x) = \sum c_n e^{i\frac{2n\pi}{L}x}.$$

Find the coefficients c_n. (**Hint**: write an expression for c_n as a sum of integrals and than change variable $y = x + nL$ and ...)