No books or notes allowed. No laptop or wireless devices allowed. Write clearly.

Name: ______________________________

| Question: | 1 | 2 | Total |
|---|---|---|---|
| Points: | 70 | 30 | 100 |
| Score: | | | |

Question 1 ........................................................................ *70 point*

The extremities of a rod of length 1 are kept at constant temperatures $T_0$ and $T_1$ while along its length it is in convective contact with a medium at a temperature that varies linearly between $T_0$ and $T_1$. This means that the temperature of the rod is governed by the equation:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} - h\left(u(x,t) - T(x)\right) \quad 0 \le x \le 1 \\ u(0,t) = T_0 \\ u(1,t) = T_1 \\ u(x,0) = \frac{T_1+T_0}{2} \end{cases} \tag{1}$$

where

$$T(x) = T_0 + (T_1 - T_0)x.$$

and $h > 0$.

(a) (15 points) Write and solve the equation for the steady state $\bar{u}(x)$ of the rod. (**Hint:** observe that $T(x)$ is a linear function that satisfies the boundary conditions.)

**Solution:**
The equation for the steady state is:

$$\begin{cases} \dfrac{\partial^2 \bar{u}(x)}{\partial x^2} - h\left(\bar{u}(x) - T(x)\right) = 0 \\ \bar{u}(0) = T_0 \\ \bar{u}(1) = T_1 \end{cases} \tag{2}$$

It is easy to see that $\bar{u}(x) = T(x)$ is a solution of the equation.

(b) (15 points) Write the equation for the deviation $v(x,t) = u(x,t) - \bar{u}(x)$.

**Solution:**

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} = \dfrac{\partial^2 v(x,t)}{\partial x^2} - hv(x,t) \quad 0 \le x \le 1 \\ v(0,t) = 0 \\ v(1,t) = 0 \\ v(x,0) = \frac{T_1-T_0}{2} - (T_1 - T_0)x \end{cases} \tag{3}$$

(c) (20 points) Use separation of variable to reduce the problem to a Sturm-Luiville problem. Find the eigenvalues and eigenfunctions.

**Solution:** Assume

$$v(x,t) = T(t)\phi(x)$$

we get

$$\dot{T}(t)\phi(x) = T(t)\phi''(x) - hT(t)\phi(x)$$

or

$$\frac{\dot{T}(t) + h}{T(t)} = \frac{\phi''(x)}{\phi(x)}$$

So that

$$\dot{T}(t) = (\lambda - h)T(t)$$

and

$$\phi''(x) = \lambda\phi(x) \qquad\qquad \phi(0) = \phi(1) = 0$$

We know that the last equation has solution:

$$\phi_n(x) = \sin(n\pi x) \qquad\qquad \lambda_n = -n^2\pi^2$$

This are the eigenvalue and eigenfunctions. The relative solution for the time function is

$$T_n(t) = e^{-(n^2\pi^2+h)t}$$

so that the general solution is

$$v(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n^2\pi^2+h)t}\sin(n\pi x)$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi} v(x,0)\sin(n\pi x)\,dx$$

(d) (20 points) Write the solution of the problem. Remember that:

$$\int x\sin(\lambda x)dx = \frac{\sin(\lambda x)}{\lambda^2} - \frac{x\cos(\lambda x)}{\lambda}$$

**Solution:** We have to compute the integrals

$$b_n = \int_0^1 \sin(n\pi x)\,dx = -\frac{\cos(n\pi x)}{n\pi}\bigg|_0^1 = \frac{1-(-1)^n}{n\pi}$$

and

$$c_n = \int_0^1 x\sin(n\pi x)\,dx = \frac{\sin(n\pi x)}{n^2\pi^2}\bigg|_0^1 - \frac{x\cos(n\pi x)}{n\pi}\bigg|_0^1 = -\frac{(-1)^n}{n\pi}.$$

so that we have

$$u(x,t) = T_0(x) + \sum_{n=1}^{\infty} a_n e^{-(n^2\pi^2+h)t}\sin(n\pi x)$$

with

$$a_0 = \frac{1+(-1)^n}{2}\frac{T_1+T_0}{n\pi}$$

Question 2 ........................................................................ *30 point*

Consider the same heat equation problem of exercise 1 but with the convection parameter $h(x)$ and the cross secrion of the rod $\rho(x)$ depending on $x$, so that the equation is now:

$$\begin{cases} \rho(x)\dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} - h(x)\left(u(x,t) - T(x)\right) & 0 \le x \le 1 \\ u(0,t) = T_0 \\ u(1,t) = T_1 \\ u(x,0) = u_0(x) \end{cases} \tag{4}$$

where $h(x)$ and $\rho(x)$ are strictly positive and continuous.

(a) (15 points) Write the Sturm-Luiville problem associated with this heat equation and state the properties of its eigenvalues and eigenfunctions relevant for the solution of the heat equation. You can use the results of part (a), (b) and (c) of exercise 1. Justify your answer.

**Solution:** Observe that the steady state $\bar{u}(x)$ is the same as in question 1). The associated Sturm-Louiville problem is now

$$\phi''(x) - h(x)\phi(x) + \mu^2\rho(x)\phi(x) = 0 \qquad \phi(0) = \phi(1) = 0$$

This is clearly a regular Sturm-Luiville problem so that we know it has infinitely many solutions $\phi_n$, $\mu_n$, $n = 1, 2, \ldots$. Moreover the $\phi_n$ are orthogonal in the sense

$$\int_0^1 \phi_n(x)\phi_m(x)\rho(x)\,dx = 0 \qquad\qquad n \neq m$$

Finally every sectionally continuous function $f(x)$ can be written as

$$f(x) = \sum_{n=1}^{\infty} a_n\phi_n(x)$$

where

$$a_n = \frac{\int_0^1 f(x)\phi_n(x)\rho(x)\,dx}{\int_0^1 \phi_n^2(x)\rho(x)\,dx}$$

and the above series converge pointwise.

(b) (15 points) Write the general solution of the heat equation in terms of the eigenvalues and eigenfunction of the Sturm-Luiville problem of point (a). Remember to give a formula, involving the initial condition, for the coefficients in the general solution.

**Solution:**

As for question 1 we have

$$u(x,t) = \bar{u}(x) + \sum_{n=1}^{\infty} a_n e^{-\mu_n^2 t} \phi_n(x)$$

where

$$a_n = \frac{\int_0^1 \big(u_0(x) - \bar{u}(x)\big)\phi_n(x)\rho(x)\,dx}{\int_0^1 \phi_n^2(x)\rho(x)\,dx}$$